ACADEMY

STUDY GUIDE

MATHEMATICS SL

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IB Academy Mathematics Study Guide

Available on learn.ib.academy

Authors: Laurence Gibbons, Robert van den Heuvel
Contributing Authors: Alex Barancova, Laurence Gibbons

Design
Rational

Typesetting

Special thanks: Vilijam Strovanovski

Version: MatSL.1.4.170418

Laan van Puntenburg 2a
3511ER, Utrecht
The Netherlands

ib.academy
info@ib.academy
+31 (0) 30 4300 430

INTRODUCTION

Welcome to the IB.Academy Study Guide for IB Mathematics Standard Level.

We are proud to present our study guides and hope that you will find them helpful. They are the result of a collaborative undertaking between our tutors, students and teachers from schools across the globe. Our mission is to create the most simple yet comprehensive guides accessible to IB students and teachers worldwide. We are firm believers in the open education movement, which advocates for transparency and accessibility of academic material. As a result, we embarked on this journey to create these study guides that will be continuously reviewed and improved. Should you have any comments, feel free to contact us.

For this Mathematics SL guide, we incorporated everything you need to know for your final exam. The guide is broken down into chapters based on the syllabus topics and they begin with 'cheat sheets' that summarise the content. This will prove especially useful when you work on the exercises. The guide then looks into the subtopics for each chapter, followed by our step-by-step approach and a calculator section which explains how to use the instrument for your exam.

For more information and details on our revision courses, be sure to visit our website at ib.academy. We hope that you will enjoy our guides and best of luck with your studies.

IB.Academy Team

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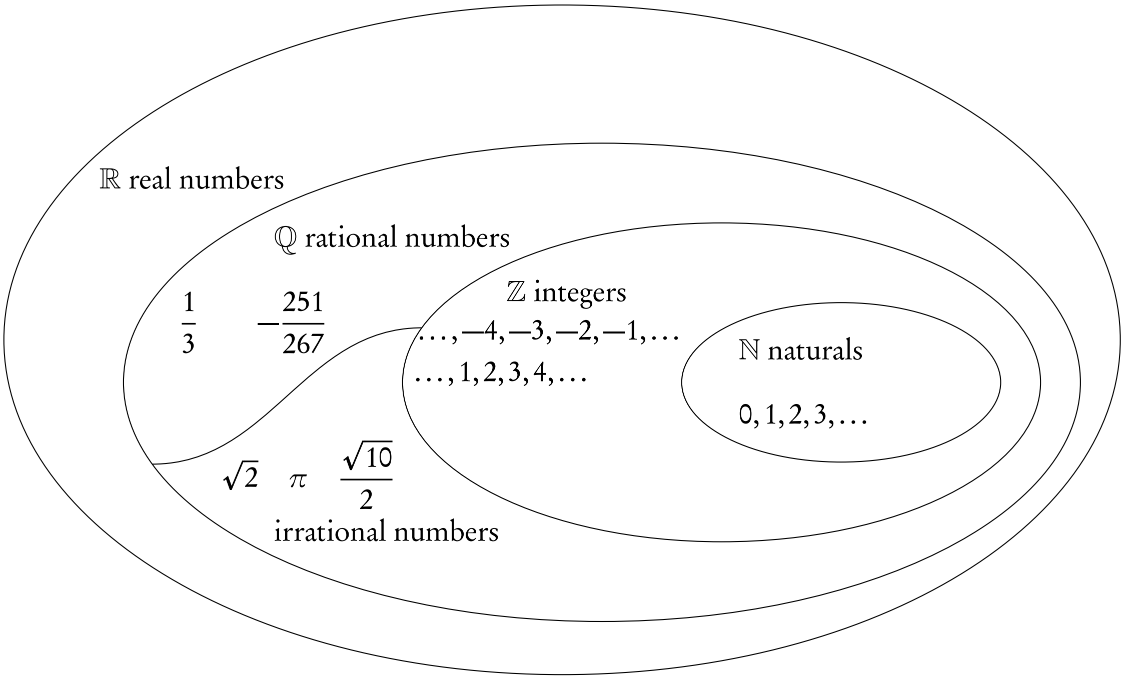
PRIOR KNOWLEDGE

1

Before you start make sure you have a firm grasp of the following. Many marks are lost through errors in these fundamentals.

1.1 Number

Numbers can be grouped in to a number of sets. From the diagram you see that all rational numbers are also real numbers; i.e. $\mathbb{Q}$ is a subset of $\mathbb{R}$.



Positive integers	$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$
Positive integers and zero	$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$
Integers	$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
Rational numbers	$\mathbb{Q} =$ any number that can be written as the ratio $\frac{p}{q}$ of any two integers, where $q \neq 0$

1.2 Signs

$+$ and $-$ signs describe positive and negative numbers. Remember they work the opposite way with negative intergers. In maths two wrongs do make a right.

$$1 - -1 = 1 + 1 = 2 \qquad -1 \times -1 = 1$$

1.3 BIDMAS

A handy acronymn for remembering the order in which to calculate equations:

B $\longrightarrow$ Brackets — functions within brackets (these should also be calculated following BIDMAS)

I $\longrightarrow$ Indecies — powers

DM $\longrightarrow$ Division/Multiplication — working from left to right

AS $\longrightarrow$ Adition/Subtraction — working from left to right

Therefore in the following equation

$$
\begin{array}{rcl}
 & & 4^2 + 5 \times \frac{6}{4} \times (9-1) = \\
\text{B} & \longrightarrow & = 4^2 + 5 \times \frac{6}{4} \times (8) = \\
\text{I} & \longrightarrow & = 16 + 5 \times \frac{6}{4} \times 8 = \\
\text{D/M} & \longrightarrow & = 16 + \frac{30}{4} \times 8 = \\
 & & = 16 + 7.5 \times 8 = \\
 & & = 16 + 60 = \\
\text{A/S} & \longrightarrow & = 76
\end{array}
$$

1.4 Solving simulatenous equations

If we have two unknows, for example x and y, and two equations, then we can solve for x and y simulateously.

$$
\begin{cases}
(1) \quad y = 3x + 1 \\
(2) \quad 2y = x - 1
\end{cases}
$$

There are 3 methods to solve simultaneous equations.

Elimination

Multiply an equation and then subtract it from the other in order to eliminate one of the unknows.

$$3 \times (2) \Rightarrow \quad (3) \quad 6y = 3x - 3$$
$$(3)-(1) \Rightarrow \quad 6y - y = 3x - 3x - 3 - 1$$
$$5y = -4$$
$$y = -\frac{4}{5}$$

Put y in (1) or (2) and solve for x

$$-\frac{4}{5} = 3x + 1$$
$$3x = -\frac{9}{5}$$
$$x = -\frac{9}{15} = -\frac{3}{5}$$

Substitution

Rearange and then substitute one in to another.

Substitute (1) into (2)

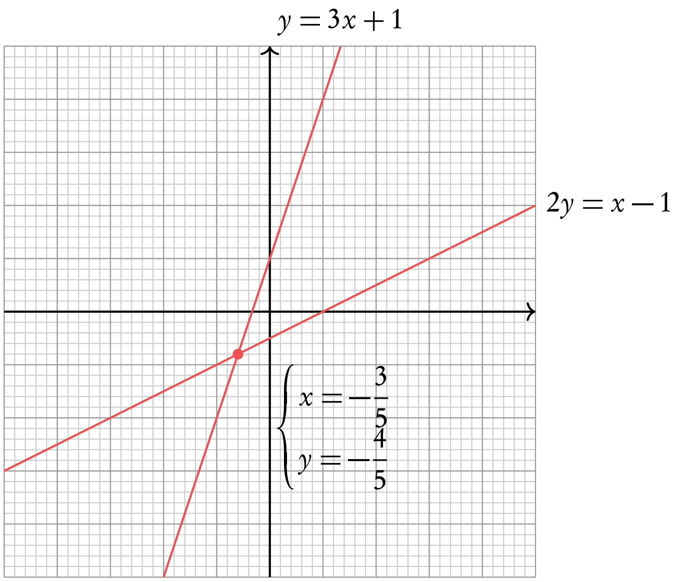
$$2(3x+1) = x - 1$$
$$6x + 2 = x - 1$$
$$5x = -3$$
$$x = -\frac{3}{5}$$

Put x in (1) or (2) and solve for x

$$y = 3(-\frac{3}{5}) + 1$$
$$y = -\frac{4}{5}$$

Graph

Graph both lines on your gdc. Where they intersect will be the solution to the equation.



Note that this method is also great when you have to solve more complex equations.

1.5 Geometry

These are given in the data booklet

Area of parallelogram	$A = \mathrm{b} \times \mathrm{h}$
Area of a triangle	$A = \frac{1}{2}(\mathrm{b} \times \mathrm{h})$
Area of a trapezium	$A = \frac{1}{2}(\mathrm{a} + \mathrm{b})\mathrm{h}$
Area of a circle	$A = \pi \mathrm{r}^2$
Circumference of a circle	$C = 2\pi \mathrm{r}$
Volume of a pyramid	$V = \frac{1}{3}(\text{area base} \times \text{vertical height})$
Volume of a cuboid (rectangular prism)	$V = \mathrm{l} \times \mathrm{w} \times \mathrm{h}$
Volume of a cylinder	$V = \pi \mathrm{r}^2 \mathrm{h}$
Area of the curved surface of a cylinder	$A = 2\pi \mathrm{rh}$
Volume of a sphere	$V = \frac{4}{3}\pi \mathrm{r}^3$
Volume of a cone	$V = \frac{1}{3}\pi \mathrm{r}^2 \mathrm{h}$

ALGEBRA

2

Table of contents & cheatsheet

2.1. Sequences — 12

Arithmetic: +/− common difference

$$u_n = n^{\text{th}} \text{ term} = u_1 + (n-1)d$$
$$S_n = \text{sum of } n \text{ terms} = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

with $u_1 = a = 1^{\text{st}}$ term, $d =$ common difference.

Geometric: ×/÷ common ratio

$$u_n = n^{\text{th}} \text{ term} = u_1 \cdot r^{n-1}$$
$$S_n = \text{sum of } n \text{ terms} = \frac{u_1(1-r^n)}{(1-r)}$$
$$S_\infty = \text{sum to infinity} = \frac{u_1}{1-r}, \text{ when } -1 < r < 1$$

with $u_1 = a = 1^{\text{st}}$ term, $r =$ common ratio.

Sigma notation
A shorthand to show the sum of a number of terms in a sequence.

$$\sum_{n=1}^{10} 3n-1$$

Last value of n

Formula

First value of n

e.g.

$$\sum_{n=1}^{10} 3n-1 = \underbrace{(3\cdot 1)-1}_{n=1} + \underbrace{(3\cdot 2)-1}_{n=2} + \cdots + \underbrace{(3\cdot 10)-1}_{n=10} = 155$$

2.2. Exponents and logarithms — 15

Exponents

$$x^1 = x \qquad x^0 = 1$$
$$x^m \cdot x^n = x^{m+n} \qquad \frac{x^m}{x^n} = x^{m-n}$$
$$(x^m)^n = x^{m\cdot n} \qquad (x\cdot y)^n = x^n \cdot y^n$$
$$x^{-1} = \frac{1}{x} \qquad x^{-n} = \frac{1}{x^n}$$
$$x^{\frac{1}{2}} = \sqrt{x} \qquad \sqrt{x}\cdot\sqrt{x} = x$$
$$\sqrt{xy} = \sqrt{x}\cdot\sqrt{y} \qquad x^{\frac{1}{n}} = \sqrt[n]{x}$$
$$x^{\frac{m}{n}} = \sqrt[n]{x^m} \qquad x^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{x^m}}$$

Logarithms

$$\log_a a^x = x \qquad a^{\log_a b} = b$$

Let $a^x = b$, isolate x from the exponent: $\log_a a^x = x = \log_a b$

Let $\log_a x = b$, isolate x from the logarithm: $a^{\log_a x} = x = a^b$

Laws of logarithms

I: $\log A + \log B = \log(A\cdot B)$

II: $\log A - \log B = \log\left(\frac{A}{B}\right)$

III: $n \log A = \log(A^n)$

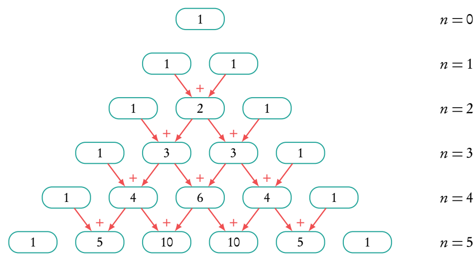
IV: $\log_B A = \frac{\log A}{\log B}$

2.3. Binomial Expansion — 17

In a expansion of a binomial in the form $(a+b)^n$. Each term can be described as $\binom{n}{r}a^n - rb^r$, where $\binom{n}{r}$ is the coefficient.
The full expansion can be written thus

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

Find the coefficient using either pascals triangle



Or the nCr function on your calculator

2.1 Sequences

2.1.1 Arithmetic sequence

Arithmetic sequence the next term is the previous number + the common difference (d).

e.g. $2, 4, 6, 8, 10, \ldots \quad d = +2$ and $2, -3, -8, -13, \ldots \quad d = -5$

To find the common difference d, subtract two consecutive terms of an arithmetic sequence from the term that follows it, i.e. $u_{(n+1)} - u_n$.

DB 1.1

Use the following equations to calculate the n^{th} term or the sum of n terms.

$$u_n = u_1 + (n-1)d \qquad S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

with

$$u_1 = a = 1^{\text{st}} \text{ term} \qquad d = \text{common difference}$$

Often the IB requires you to first find the 1st term and/or common difference.

Finding the first term u_1 and the common difference d from other terms.

In an arithmetic sequence $u_{10} = 37$ and $u_{22} = 1$. Find the common difference and the first term.

1.	Put numbers in to n^{th} term formula	$37 = u_1 + 9d$ $1 = u_1 + 21d$
2.	Equate formulas to find d	$21d - 1 = 9d - 37$ $12d = -36$ $d = -3$
3.	Use d to find u_1	$1 - 21 \cdot (-3) = u_1$ $u_1 = 64$

2.1.2 Geometric sequence

Geometric sequence the next term is the previous number multiplied by the common ratio (r).

To find the common ratio, divide any term of an arithmetic sequence by the term that precedes it, i.e. $\dfrac{\text{second term } (u_2)}{\text{first term } (u_1)}$ e.g. $2, 4, 8, 16, 32, \ldots \quad r = 2$

and $25, 5, 1, 0.2, \ldots \quad r = \dfrac{1}{5}$

Use the following equations to calculate the n^{th} term, the sum of n terms or the sum to infinity when $-1 < r < 1$. DB 1.1

$$u_n = n^{\text{th}} \text{ term} \qquad S_n = \text{sum of } n \text{ terms} \qquad S_\infty = \text{sum to infinity}$$

$$= u_1 \cdot r^{n-1} \qquad = \frac{u_1(1-r^n)}{(1-r)} \qquad = \frac{u_1}{1-r}$$

again with

$$u_1 = a = 1^{\text{st}} \text{ term} \qquad r = \text{common ratio}$$

Similar to questions on Arithmetic sequences, you are often required to find the 1st term and/or common ratio first.

2.1.3 Sigma notation

Sigma notation is a way to represent the summation of any sequence — this means that it can be used for both arithmetic or geometric series. The notation shows you the formula that generates terms of a sequence and the upper and lower limits of the terms that you want to add up in this sequence.

Last value of n

$$\sum_{n=1}^{10} 3n - 1 \longleftarrow \text{Formula}$$

First value of n

e.g. $\displaystyle\sum_{n=1}^{10} 3n - 1 = \underbrace{(3\cdot 1) - 1}_{n=1} + \underbrace{(3\cdot 2) - 1}_{n=2} + \underbrace{(3\cdot 3) - 1}_{n=3} + \cdots + \underbrace{(3\cdot 10) - 1}_{n=10} = 155$

Finding the first term u_1 and common ratio r from other terms.	
$\sum_{1}^{5}(\text{Geometric series}) = 3798,\ \sum_{1}^{\infty}(\text{Geometric series}) = 4374.$ Find $\sum_{1}^{7}(\text{Geometric series}) = ?$	
1. Interpret the question	The sum of the first 5 terms of a geometric sequence is 3798 and the sum to infinity is 4374. Find the sum of the first 7 terms
2. Use formula for sum of n terms	$3798 = u_1 \dfrac{1-r^5}{1-r}$
3. Use formula for sum to infinity	$4374 = \dfrac{u_1}{1-r}$
4. Rearrange **3.** for u_1	$4374(1-r) = u_1$
5. Substitute in to **2.**	$3798 = \dfrac{4374(1-r)\left(1-r^5\right)}{1-r}$
6. Solve for r	$3798 = 4374\left(1-r^5\right)$ $\dfrac{3798}{4374} = 1-r^5$ $r^5 = 1-\dfrac{211}{243}$ $\sqrt[5]{r} = \sqrt[5]{\dfrac{32}{243}}$ $r = \dfrac{2}{3}$
7. Use r to find u_1	$u_1 = 4374\left(1-\dfrac{2}{3}\right)$ $u_1 = 1458$
8. Find sum of first 7 terms	$1458\dfrac{1-\left(\frac{2}{3}\right)^7}{1-\frac{2}{3}} = 4370$

ACADEMY

2.2 Exponents and logarithms

2.2.1 Laws of exponents

Exponents always follow certain rules. If you are multiplying or dividing, use the following rules to determine what happens with the powers.

Example.

$$x^1 = x \qquad 6^1 = 6$$
$$x^0 = 1 \qquad 7^0 = 1$$
$$x^m \cdot x^n = x^{m+n} \qquad 4^5 \cdot 4^6 = 4^{11}$$
$$\frac{x^m}{x^n} = x^{m-n} \qquad \frac{3^5}{3^4} = 3^{5-4} = 3^1 = 3$$
$$(x^m)^n = x^{m \cdot n} \qquad \left(10^5\right)^2 = 10^{10}$$
$$(x \cdot y)^n = x^n \cdot y^n \qquad (2 \cdot 4)^3 = 2^3 \cdot 4^3 \quad \text{and} \quad (3x)^4 = 3^4 x^4$$
$$x^{-1} = \frac{1}{x} \qquad 5^{-1} = \frac{1}{5} \quad \text{and} \quad \left(\frac{3}{4}\right)^{-1} = \frac{4}{3}$$
$$x^{-n} = \frac{1}{x^n} \qquad 3^{-5} = \frac{1}{3^5} = \frac{1}{243}$$

2.2.2 Fractional exponents

When doing mathematical operations (+, −, × or ÷) with fractions in the exponent you will need the following rules. These are often helpful when writing your answers in simplest terms.

Example.

$$x^{\frac{1}{2}} = \sqrt{x} \qquad 2^{\frac{1}{2}} = \sqrt{2}$$
$$\sqrt{x} \cdot \sqrt{x} = x \qquad \sqrt{3} \cdot \sqrt{3} = 3$$
$$\sqrt{xy} = \sqrt{x} \cdot \sqrt{y} \qquad \sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2 \cdot \sqrt{3}$$
$$x^{\frac{1}{n}} = \sqrt[n]{x} \qquad 5^{\frac{1}{3}} = \sqrt[3]{5}$$
$$x^{\frac{m}{n}} = \sqrt[n]{x^m} \qquad 3^{-\frac{2}{5}} = \frac{1}{\sqrt[5]{3^2}}$$

2.2.3 Laws of logarithms

DB 1.2

Logarithms are the inverse mathematical operation of exponents, like division is the inverse mathematical operation of multiplication. The logarithm is often used to find the variable in an exponent.

$$a^x = b \quad \Leftrightarrow \quad x = \log_a b$$

Since $\log_a a^x = x$, so that $x = \log_a b$.

This formula shows that the variable x in the power of the exponent becomes the subject of your log equation, while the number a becomes the base of your logarithm.

Below are the rules that you will need to use when performing calculations with logarithms and when simplifying them. The sets of equations on the left and right are the same; on the right we show the notation that the DB uses while the equations on the left are easier to understand.

Laws of logarithms and change of base

DB 1.2

I:	$\log A + \log B = \log(A \cdot B)$	$\log_c a + \log_c b = \log_c(ab)$
II:	$\log A - \log B = \log\left(\frac{A}{B}\right)$	$\log_c a - \log_c b = \log_c\left(\frac{a}{b}\right)$
III:	$n \log A = \log(A^n)$	$n \log_c a = \log_c(a^n)$
IV:	$\log_B A = \frac{\log A}{\log B}$	$\log_b a = \frac{\log_c a}{\log_c b}$

Note

- $x = \log_a a = 1$
- With the 4^{th} rule you can change the base of a log.
- $\log_a 0 = x$ is always undefined (because $a^x \neq 0$).
- When you see a log with no base, it is referring to a logarithm with a base of 10 (e.g. $\log 13 = \log_{10} 13$).

Solve x in exponents using logs.

Solve $2^x = 13$.

1.	Take the log on both sides	$\log 2^x = \log 13$
2.	Use rule III to take x outside	$x \log 2 = \log 13$
3.	Solve	$x = \frac{\log 13}{\log 2}$

Expressing logs in terms of other logs

Example.

For the following three examples use that $p = \log_a 5$ and $q = \log_a 2$.

1. Express $\log_a 10$ in terms of p and q: $\log_a(5 \times 2) = \log_a 5 + \log_a 2 = p + q$
2. Express $\log_a 8$ in terms of p and q: $\log_a \left(2^3\right) = 3\log_a 2 = 3q$
3. Express $\log_a 2.5$ in terms of p and q: $\log_a \left(\frac{5}{2}\right) = \log_a 5 - \log_a 2 = p - q$

But what about ln and e? These work exactly the same; e is *just* the irrational number 2.71828... (infinitely too long to write out) and ln is just $\log_e$.

$$\ln a + \ln b = \ln(a \cdot b)$$
$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$
$$n \ln a = \ln a^n$$
$$\ln e = 1$$
$$e^{\ln a} = a$$

2.3 Binomial expansion

Binomial an expression $(a+b)^n$ which is the sum of *two terms* raised to the power n.

e.g. $(x+3)^2$

Binomial expansion $(a+b)^n$ is expanded into a sum of terms

e.g. $x^2 + 6x + 9$

Binomial expansions get increasingly complex as the power increases:

binomial	**binomial expansion**
$(a+b)^1$	$= a + b$
$(a+b)^2$	$= a^2 + 2ab + b^2$
$(a+b)^3$	$= a^3 + 3a^2b + 3ab^2 + b^3$

The general formula for each term is: $\binom{n}{r} a^{n-r} b^r$.

In order to find the full binomial expansion of a binomial, you have to determine *the coefficient* $\binom{n}{r}$ and *the powers* for each term, $n - r$ and r for a and b respectively, as shown by the binomial expansion formula.

Binomial expansion formula

DB 1.3

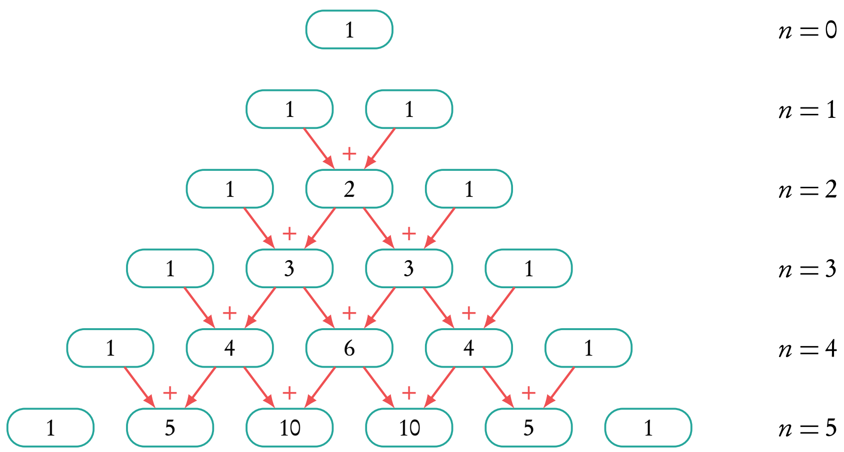
$$
\begin{aligned}
(a+b)^n &= a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n \\
&= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots
\end{aligned}
$$

The powers decrease by 1 for a and increase by 1 for b for each subsequent term.

The sum of the powers of each term will always $= n$.

There are two ways to find the coefficients: with Pascal's triangle or the binomial coefficient function (nCr).

Pascal's triangle



Pascal's triangle is an easy way to find *all* the coefficients for your binomial expansion. It is particularly useful in cases where:

1. the power is not too high (because you have to write it out manually);
2. if you have to find all the terms in a binomial expansion.

Binomial coefficient functions

In the 1st term of the expansion $r = 0$, in the 2nd term $r = 1, \ldots$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Use the nCr function on your calculator!

How to expand binomial expansions.

Find the expansion of $\left(x-\frac{2}{x}\right)^5$

1. Use the binomial expansion formula

$a = x$

$b = -\frac{2}{x}$

$$(x)^5 + (5C1)(x)^4\left(-\frac{2}{x}\right) + (5C2)(x)^3\left(-\frac{2}{x}\right)^2 + (5C3)(x)^2\left(-\frac{2}{x}\right)^3 + (5C4)(x)\left(-\frac{2}{x}\right)^4 + (5C5)\left(-\frac{2}{x}\right)^5$$

2. Find coefficients

using Pascal's triangle for low powers and nCr calculator for high functions

Row 0:			1			
Row 1:			1	1		
Row 2:		1	2	1		
Row 3:		1	3	3	1	
Row 4:	1	4	6	4	1	
Row 5:	1	5	10	10	5	1

$(5C0)=1$, $(5C1)=5$, $(5C2)=10$, $(5C3)=10$, $(5C4)=5$, $(5C5)=1$

3. Put them together

$$x^5 + 5x^4\left(-\frac{2}{x}\right)^1 + 10x^3\left(-\frac{2}{x}\right)^2 + 10x^2\left(-\frac{2}{x}\right)^3 + 5x\left(-\frac{2}{x}\right)^4 + \left(-\frac{2}{x}\right)^5$$

4. Simplify

using laws of exponents

$$x^5 - 10x^3 + 20x - \frac{80}{x} + \frac{80}{x^3} - \frac{32}{x^5}$$

The IB use three different terms for these types of question which will effect the answer you should give:

Coefficient: the number before the x value;

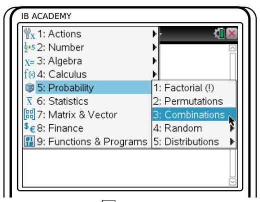
Term: the number and the x value;

Constant term: the number for which there is no x value (x^0).

Finding a specific term in a binomial expansion.

Find the coefficient of x^5 in the expansion $(2x-5)^8$

1.	One term is asked, usually of a high power then use binomial expansion formula	$(a+b)^n = \cdots + \binom{n}{r} a^{n-r} b^r + \ldots$
2.	Determine r	Since $a = 2x$, to find x^5 we need a^5. $a^5 = a^{n-r} = a^{8-r}$, so that $r = 3$
3.	Plug r into the general formula	$\binom{n}{r} a^{n-r} b^r = \binom{8}{3} a^{8-3} b^3 = \binom{8}{3} a^5 b^3$
4.	Replace a and b	$\binom{8}{3}(2x)^5(-5)^3$
5.	Use nCr to calculate the value for $\binom{n}{r}$	$\binom{8}{3} = 8C3 = 56$



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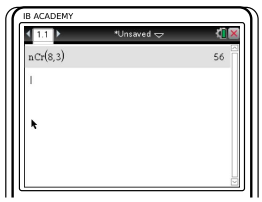
3: Combinations



Insert the values for n and r separated by a comma

6.	Substitute and calculate the value	$56 \times 2^5(x^5) \times (-5)^3 = -224000(x^5)$

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FUNCTIONS

3

Table of contents & cheatsheet

Definitions

Function a mathematical relationship where each input has a single output. It is oft en written as $f(x)$ where x is the input
Domain all possible x values, the input. (the domain of investigation)
Range possible y values, the output. (the range of outcomes)
Coordinates uniquely determines the position of a point, given by (x, y)

3.1. Types of functions 22

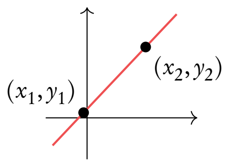
Linear functions $y = mx + c$

m is the *gradient*,
c is the y *intercept*.

Midpoint: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Distance: $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Gradient: $m = \frac{y_2-y_1}{x_2-x_1}$

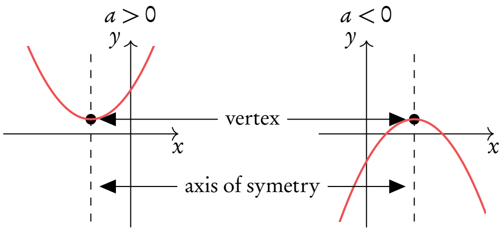


Parallel lines: $m_1 = m_2$ (same gradients)
Perpendicular lines: $m_1 m_2 = -1$

Quadratic functions $y = ax^2 + bx + c = 0$

Axis of symmetry: x-coordinate of the vertex: $x = \frac{-b}{2a}$

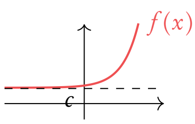
Factorized form: $y = (x+p)(x+q)$



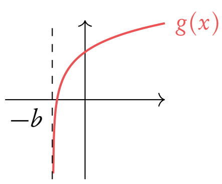
If $a = 1$ use the factorization method $(x+p)\cdot(x+q)$
If $a \neq 1$ use the quadratic formula
When asked excplicity complete the square

Vertex form: $y = a(x-h)^2 + k$
Vertex: (h, k)

Exponential
$f(x) = a^x + c$

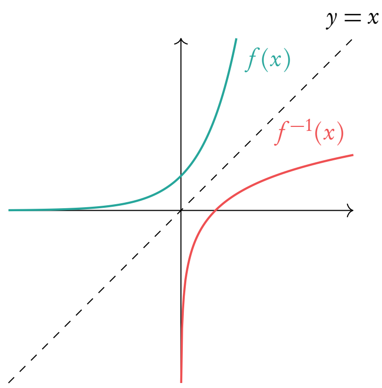


Logarithmic
$g(x) = \log_a(x+b)$



3.2. Rearranging functions 26

Inverse function, $f^{-1}(x)$ reflection of $f(x)$ in $y = x$.



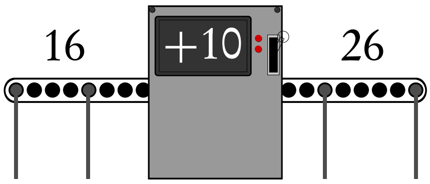
Composite function, $(f \circ g)(x)$ is the combined function f of g of x.
When $f(x)$ and $g(x)$ are given, replace x in $f(x)$ by $g(x)$.

Transforming functions

Change to $f(x)$	Effect
$f(x)+a$	Move graph a units upwards
$f(x+a)$	Move graph a units to the left
$a \cdot f(x)$	Vertical stretch by factor a
$f(a \cdot x)$	Horizontal stretch by factor $\frac{1}{a}$
$-f(x)$	Reflection in x-axis
$f(-x)$	Reflection in y-axis

3.1 Types of functions

Functions are mathematical relationships where each input has a single output. You have probably been doing functions since you began learning maths, but they may have looked like this:



Algebraically this is:
$f(x) = x + 10$,
here $x = 16$, $y = 26$.

We can use graphs to show multiple outputs of y for inputs x, and therefore visualize the relation between the two. Two common types of functions are linear functions and quadratic functions.

3.1.1 Linear functions

Linear functions $y = mx + c$ increases/decreases at a constant rate m, where m is the *gradient* and c is the y *intercept*.

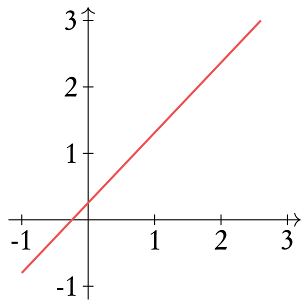
Midpoint $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Distance $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Gradient $m = \frac{y_2-y_1}{x_2-x_1}$

Parallel lines $m_1 = m_2$ (equal gradients)

Perpendicular lines $m_1 m_2 = -1$



Example.

Determine the midpoint, distance and gradient using the two points $P_1(2,8)$ and $P_2(6,3)$

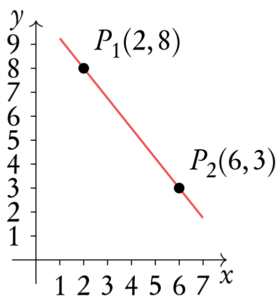
Midpoint: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{2+6}{2}, \frac{8+3}{2}\right) = (4, 5.5)$

Distance: $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(6-2)^2+(3-8)^2} = \sqrt{(4)^2+(5)^2} = \sqrt{41}$

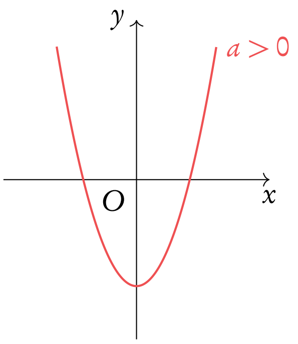
Gradient: $m = \frac{y_2-y_1}{x_2-x_1} = m = \frac{3-8}{6-2} = -\frac{5}{4}$

Parallel line: $-\frac{5}{4}x + 3$

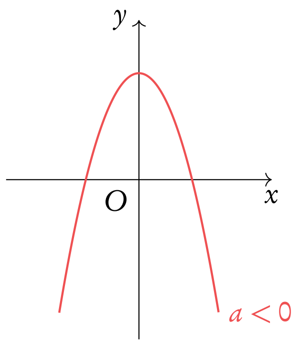
Perpendicular line: $-\frac{4}{5}x + 7$



3.1.2 Quadratic functions



$a > 0$, positive quadratic



$a < 0$, negative quadr

Quadratic functions $y = ax^2 + bx + c = 0$

Graph has a parabolic shape, increase/decrease at an increasing rate.

The roots of an equation are the x-values for which $y = 0$, in other words the x-intercept(s).

To find the roots of the equation you can use

factorisation: If $a = 1$, use the factorization method $(x + p) \cdot (x + q)$

quadratic formula: If $a \neq 1$, use the quadratic formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a}$$

The $b^2 - 4ac$ part of the quadratic formula is also known as the discriminant Δ. It can be used to check how many x-intercepts the equation has:
$\Delta > 0$: 2 solutions
$\Delta = 0$: 1 solution
$\Delta < 0$: no real solutions

Solving quadratic equations by factorisation.

Solve: $x^2 - 5x + 6 = 0$

1.	Set up system of equations $p + q = b$ and $p \times q = c$	$\left.\begin{array}{l} p+q=-5 \\ p \times q = 6 \end{array}\right\} p = -2$ and $q = -3$
2.	Plug the values for p and q into: $(x+p)(x+q)$	$(x-2)(x-3) = x^2 - 5x + 6$
3.	Equate each part to 0 $(x+p) = 0$, $(x+q) = 0$, and solve for x	$\left.\begin{array}{l} (x-2)=0 \\ (x-3)=0 \end{array}\right\} x = 2$ or $x = 3$

ACADEMY

	Solving quadratic equations using the quadratic formula.	
	Solve: $3x^2 - 8x + 4 = 0$	
1.	Calculate the discriminant Δ $\Delta = b^2 - 4ac$	$\Delta = (-8)^2 - 4 \cdot 3 \cdot 4 = 16$
2.	How many solutions? $\Delta > 0 \Rightarrow$ 2 solutions $\Delta = 0 \Rightarrow$ 1 solution $\Delta < 0 \Rightarrow$ no real solutions	$\Delta > 0$, so 2 solutions
3.	Calculate x, use $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$	$x = \dfrac{8 \pm \sqrt{16}}{2 \cdot 3} = \dfrac{8 \pm 4}{6}$ $\left.\begin{aligned} &= \dfrac{8-4}{6} = \dfrac{4}{6} \\ &= \dfrac{8+4}{6} = 2 \end{aligned}\right\} \Rightarrow \quad x = \dfrac{2}{3}$ or $x = 2$

By completing the square you can find the value of the *vertex* (the minimum or maximum). For the exam you will always be asked explicitly.

	Find the vertex by completing the square	
	$4x^2 - 2x - 5 = 0$	
1.	Move c to the other side	$4x^2 - 2x = 5$
2.	Divide by a	$x^2 - \dfrac{1}{2}x = \dfrac{5}{4}$
3.	Calculate $\left(\dfrac{x \text{ coeficient}}{2}\right)^2$	$\left(\dfrac{-\frac{1}{2}}{2}\right)^2 = \dfrac{1}{16}$
4.	Add this term to both sides	$x^2 - \dfrac{1}{2}x + \dfrac{1}{16} = \dfrac{5}{4} + \dfrac{1}{16}$
5.	Factor perfect square, bring constant back	$\left(x - \dfrac{1}{4}\right)^2 - \dfrac{21}{16} = 0$ $\Rightarrow$ minimum point$=\left(\dfrac{1}{4}, -\dfrac{21}{16}\right)$

Other forms: $y = a(x-h)^2 + k$ vertex (h,k) and $y = a(x-p)(x-q)$, x intercepts: $(p,0)(q,0)$.

3.1.3 Functions with asymptotes

Asymptote a straight line that a curve approaches, but never touches.

A single function can have multiple asymptotes: horizontal, vertical and in rare cases diagonal. Functions that contain the variable (x) in the denominator of a fraction will always have asymptotes, as well as exponential and logarithmic functions.

Vertical asymptotes

Vertical asymptotes occur when the denominator is zero, as dividing by zero is undefinable. Therefore if the denominator contains x and there is a value for x for which the denominator will be 0, we get a vertical asymptote.

Example.

In the function $f(x) = \dfrac{x}{x-4}$, when $x = 4$, the denominator is 0 so there is a vertical asymptote.

Horizontal asymptotes

Horizontal asymptotes are the value that a function tends to as x become really big or really small; technically: to the limit of infinity, $x \to \infty$. When x is large other parts of the function not involving x become insignificant and so can be ignored.

Example.

In the function $f(x) = \dfrac{x}{x-4}$, when x is small the 4 is important.

$x = 10$ $\qquad$ $10 - 4 = 6$

But as x gets bigger the 4 becomes increasingly insignificant

$x = 100$ $\qquad$ $100 - 4 = 96$

$x = 10000$ $\qquad$ $10000 - 4 = 9996$

Therefore as we approach the limits we can ignore the 4.

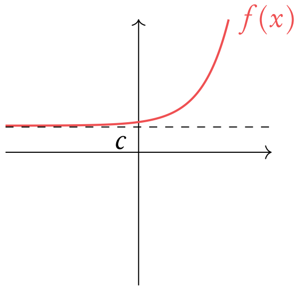
$$\lim_{x \to \infty} f(x) = \frac{x}{x} = 1$$

So there is a horizontal asymptote at $y = 1$.

Exponential and logarithmic functions

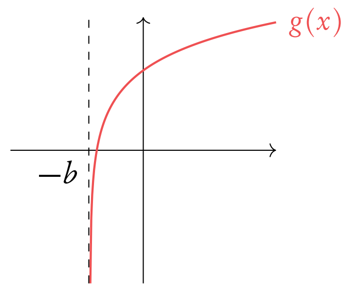
Exponential functions will always have a horizontal asymptote and logarithmic functions will always have a vertical asymptote, due to the nature of these functions. The position of the asymptote is determined by constants in the function.

Exponential



$f(x) = a^x + c$
where a is a positive number (often e)

Logarithmic

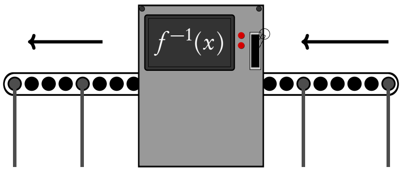


$g(x) = \log_a(x + b)$

3.2 Rearranging functions

3.2.1 Inverse functions, $f^{-1}(x)$

Inverse functions are the reverse of a function. Finding the input x for the output y. You can think of it as going backwards through the number machine



This is the same as reflecting a graph in the $y = x$ axis.

Finding the inverse function.

$f(x) = 2x^3 + 3$, find $f^{-1}(x)$

1.	Replace $f(x)$ with y	$y = 2x^3 + 3$
2.	Solve for x	$y - 3 = 2x^3$ $\Rightarrow \dfrac{y-3}{2} = x^3$ $\Rightarrow \sqrt[3]{\dfrac{y-3}{2}} = x$
3.	Replace x with $f^{-1}(x)$ and y with x	$\sqrt[3]{\dfrac{x-3}{2}} = f^{-1}(x)$

3.2.2 Composite functions

Composite functions are combination of two functions.

$$(f \circ g)(x) \qquad \text{means } f \text{ of } g \text{ of } x$$

To find the composite function above substitute the function of $g(x)$ into the x of $f(x)$.

Example.

Let $f(x) = 2x + 3$ and $g(x) = x^2$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

$(f \circ g)(x)$: replace x in the $f(x)$ function with the entire $g(x)$ function

$$(2g(x)) + 3 = 2x^2 + 3$$

$(g \circ f)(x)$: replace x in the $g(x)$ function with the entire $f(x)$ function

$$\big(f(x)\big)^2 = (2x+3)^2$$

3.2.3 Transforming functions

By adding and/or multiplying by constants we can transform a function into another function.

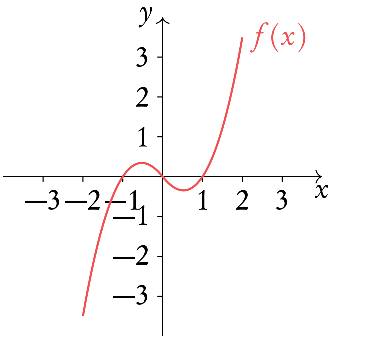
Change to $f(x)$	Effect
$f(x)+a$	Move graph a units upwards
$f(x+a)$	Move graph a units to the left
$a \cdot f(x)$	Vertical stretch by factor a
$f(a \cdot x)$	Horizontal stretch by factor $1/a$
$-f(x)$	Reflection in x-axis
$f(-x)$	Reflection in y-axis

Exam hint: describe the transformation with words as well to guarantee marks.

Transforming functions $f(x) \to af(x+b)$

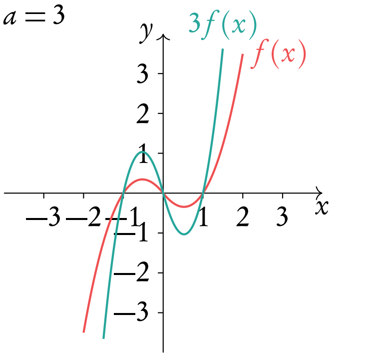
Given $f(x) = \frac{1}{4}x^3 + x^2 - \frac{5}{4}x$, draw $3f(x-1)$.

1. Sketch $f(x)$



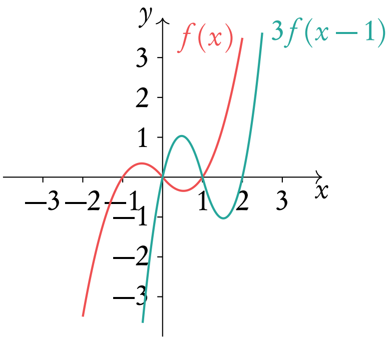
2. Stretch the graph by the factor of a

$a = 3$



3. Move graph by $-b$

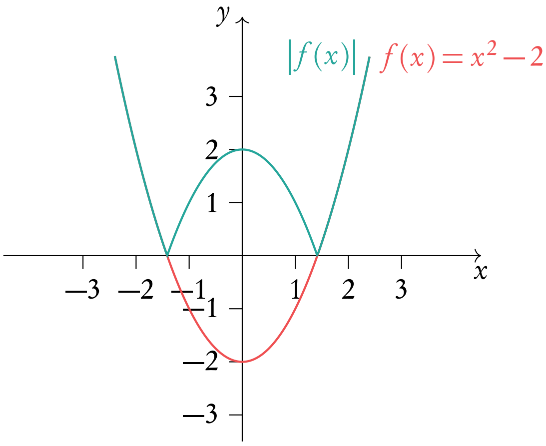
Move graph by 1 to the *right*



Example.

Absolute value: $|f|$

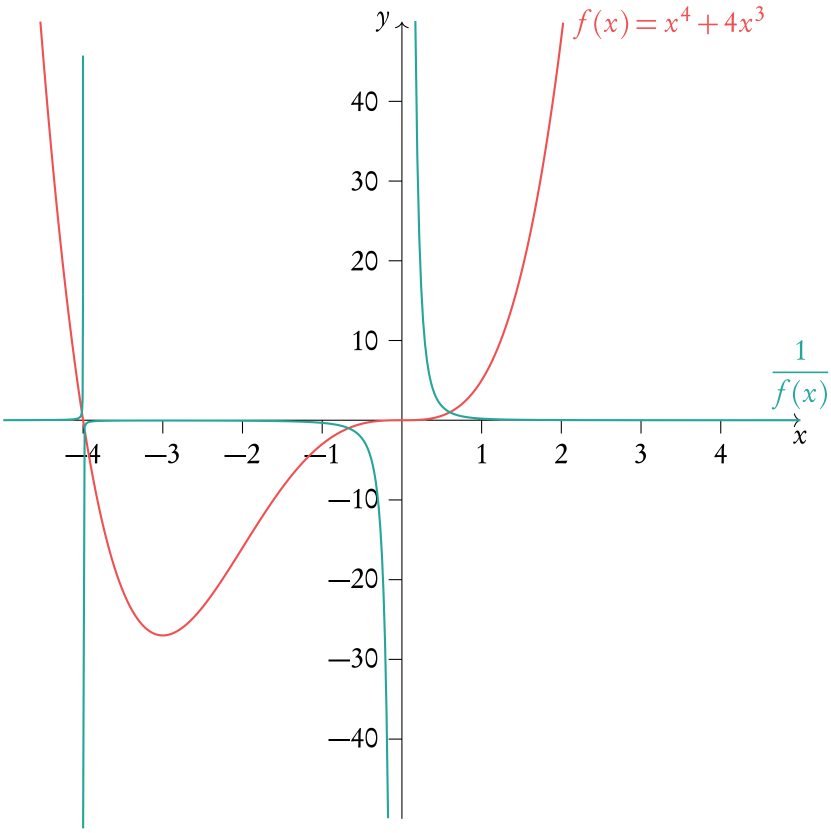
$f(x) = x^2 - 2 \Rightarrow |f(x)| =?.$



Example.

Reciprocal: $\frac{1}{f(x)}$

$f(x) = x^4 + 4x^3$ so: $\frac{1}{f(x)} = \frac{1}{x^4 + 4x^3}$



VECTORS

4

Table of contents & cheatsheet

Definitions

Vector a geometric object with *magnitude* (length) and *direction*, represented by an *arrow*.

Collinear points points that lie on the same line

Unit vector vector with magnitude 1

Base vector $\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$

4.1. Working with vectors 30

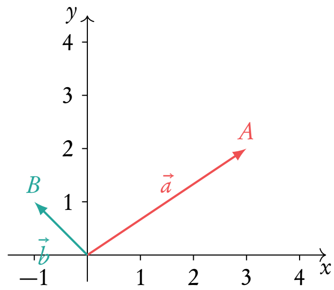
Vector from point O to point A: $\vec{OA} = \vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Vector from point O to point B: $\vec{OB} = \vec{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Can be written in two ways:

$\vec{a} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\vec{a} = 3i + 2j + 0k = 3i + 2j$



Length of $\vec{a}$: $|\vec{a}| = \sqrt{x^2 + y^2} = \sqrt{3^2 + 2^2} = \sqrt{13}$

Addition & multiplication: $\vec{a} + 2\vec{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

Subtraction: $\vec{a} - \vec{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

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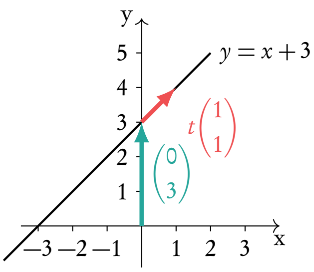
Example of a line:

$r = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

direction vector

parameter

position vector



4.3. Dot product 34

The dot product of two vectors $\vec{c} \cdot \vec{d}$ can be used to find the angle between them.

Let $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \vec{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$:

$$\vec{c} \cdot \vec{d} = |\vec{c}||\vec{d}| \cos\theta$$

$$\vec{c} \cdot \vec{d} = c_1 d_1 + c_2 d_2 + c_3 d_3$$

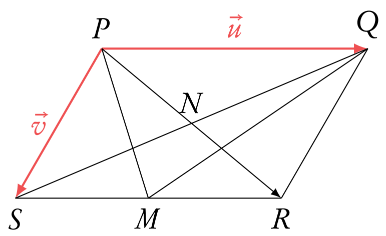
ACADEMY

4.1 Working with vectors

Vectors are a geometric object with a *magnitude* (length) and *direction*. They are represented by an *arrow*, where the arrow shows the direction and the length represents the magnitude.

So looking at the diagram we can see that vector $\vec{u}$ has a greater magnitude than $\vec{v}$. Vectors can also be described in terms of the points they pass between. So

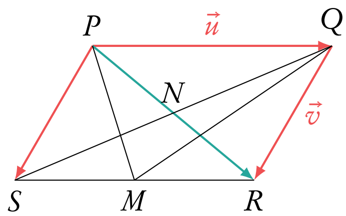
$$\begin{cases} \vec{u} = \vec{PQ} \\ \vec{v} = \vec{PS} \end{cases}$$

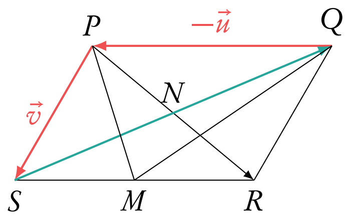


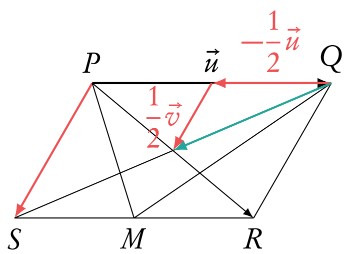
with the arrow over the top showing the direction.

You can use vectors as a geometric algebra, expressing other vectors in terms of $\vec{u}$ and $\vec{v}$. For example

$$\vec{PR} = \vec{u} + \vec{v} \qquad \vec{QS} = -\vec{u} + \vec{v} \qquad \vec{QN} = \frac{1}{2}(-\vec{u} + \vec{v})$$







This may seem slightly counter-intuitive at first. But if we add in some possible figures you can see how it works. If $\vec{u}$ moves 5 units to the left and $\vec{v}$ moves 1 unit to the right (−left) and 3 units down.

Then $\vec{PR} = \vec{u} + \vec{v} =$ 5 units to the left −1 unit to the right and 3 units down = 4 units to the left and 3 units down.

4.1.1 Vectors with value

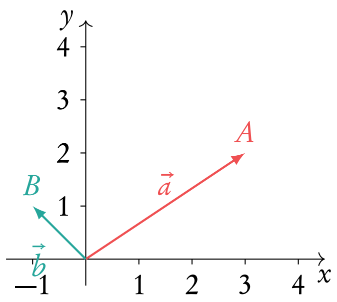
Formally the value of a vector is defined by its direction and magnitude within a 2D or 3D space. You can think of this as the steps it has to take to go from its starting point to its end, moving only in the x, y and z axis.

Vector from point O to point A:

$$\vec{OA} = \vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Vector from point O to point B:

$$\vec{OB} = \vec{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



Vectors can be written in two ways:

Note: unless told otherwise, answer questions in the form used in the question.

1. $\vec{a} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, where the top value is movement in the x-axis. Then the next is movement in the y and finally in the z. Here the vector is in 2D space as there is no value for the z-axis.
2. as the sum of the three base vectors:

$$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Here $\vec{i}$ is moving 1 unit in the x-axis, $\vec{j}$ 1 unit in the y-axis and $\vec{k}$ 1 unit in the z-axis.

$$\vec{a} = 3i + 2j + 0k = 3i + 2j$$

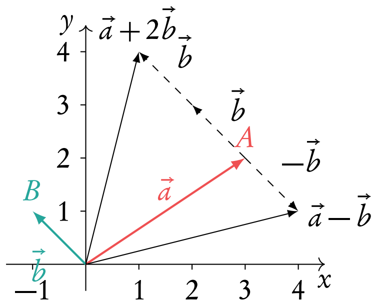
When we work with vectors we carry out the mathematical operation in each axis separately. So x-values with x-values and so on.

Addition & multiplication:

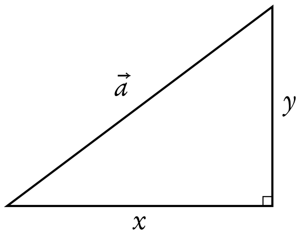
$$\vec{a} + 2\vec{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Subtraction:

$$\vec{a} - \vec{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$



However it must be remembered that vector notation does not give us the actual length (magnitude) of the vector. To find this we use something familiar.



Length of $\vec{a}$:

$$
\begin{aligned}
|\vec{a}| &= \sqrt{x^2 + y^2} \\
&= \sqrt{3^2 + 2^2} = \sqrt{13}
\end{aligned}
$$

Sometimes you will be asked to work with unit vectors. These are vectors with a magnitude of 1. We can convert all vectors to unit vectors.

Determine the unit vector $\hat{a}$ in the direction of any vector $\vec{a}$

$$
\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{3}{\sqrt{13}}\vec{i} + \frac{2}{\sqrt{13}}\vec{j} = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ 2 \end{pmatrix}
$$

4.2 Equations of lines

We can further divide vectors into two types:

position vectors vectors from the origin to a point,

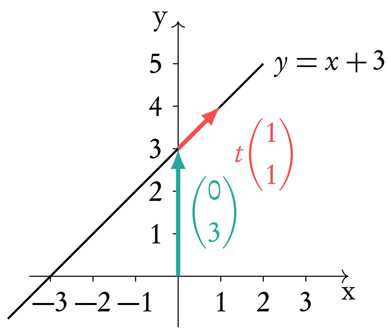
e.g. $P = (-1, 3) \Rightarrow \vec{P} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

direction vectors vectors that define a direction.

Using both we can define lines in terms of vectors.
Example of a line:

$$
r = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix}
$$

direction vector
parameter
position vector



Note the position vector can go to any where on the line. So in this example we could also use $(-3, 0)$ or $(1, 4)$. Equally the direction vector can be scaled. So we could used $(2, 2)$, $(30, 30)$, ...

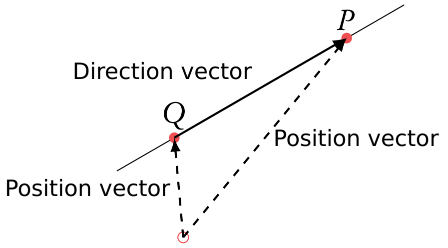
Because of this parallel lines will have direction vectors with the same ratio but not necessarily in exact numbers.

Parallel lines: direction vector of L_1 = direction vector of L_2 × constant

Questions often deal with points and or multiple lines. It is worth making a sketch to help understand the question.

Finding a line passing through two points.

Find the equation of the line passing through points $P=(1,3,2)$ and $Q=(0,-1,4)$. Does point $R=(-2,9,1)$ lie on the line?



Note this can go either way from Q to P of P to Q.

1. Write points as position vectors

$$\vec{P}=\begin{pmatrix}1\\3\\2\end{pmatrix},\ \vec{Q}=\begin{pmatrix}0\\-1\\4\end{pmatrix}$$

2. Direction vector
= vector between points

$$\begin{pmatrix}0-1\\-1-3\\4-2\end{pmatrix}=\begin{pmatrix}-1\\-4\\2\end{pmatrix}$$

3. Choose $\vec{P}$ or $\vec{Q}$ as position vector

$$r=\begin{pmatrix}1\\3\\2\end{pmatrix}+t\begin{pmatrix}-1\\-4\\2\end{pmatrix}$$

4. Equate $\vec{R}$ and the line r.
If there is no contradiction,
R lies on r

$$\begin{pmatrix}-2\\9\\1\end{pmatrix}=\begin{pmatrix}1\\3\\2\end{pmatrix}+t\begin{pmatrix}-1\\-4\\2\end{pmatrix}$$

$\Rightarrow -2=1-t \Rightarrow t=3$

$\Rightarrow 9=3-4t \Rightarrow 9\neq 3-12$

$\Rightarrow R$ does not lie on the line.

Finding the intersection of two lines.

Find the intersection for $r_1=\begin{pmatrix}2\\1\\0\end{pmatrix}+s\begin{pmatrix}-3\\1\\4\end{pmatrix}$ and $r_2=\begin{pmatrix}-1\\3\\7\end{pmatrix}+t\begin{pmatrix}3\\0\\-1\end{pmatrix}$

1. Equate
write simultaneous equations

$$\begin{cases}2-3s=-1+3t\\1+s=3\end{cases}$$

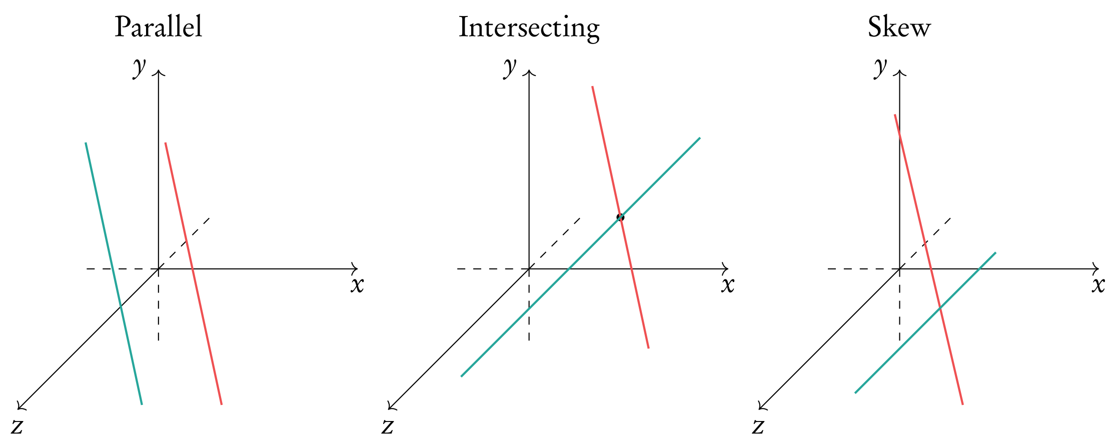
2. Solve

$s=2,\ t=-1$

3. Substitute back into r_1 or r_2

$$\begin{pmatrix}2-3(2)\\1+2\\4(2)\end{pmatrix}=\begin{pmatrix}-4\\3\\8\end{pmatrix}$$

If one considers two lines in a three-dimensional graph, then there are three ways in which they can interact:



If direction vectors defining a line aren't multiples of one another, then the lines can either be intersectiong or skew. One can find out if the lines intersect by equationg the vector equations and attempting to solve the set of equations (remember: one needs as many equations as variable to solve).

If one can't find a point of intersection, then the lines are skew.

4.3 Dot (scalar) product

DB 4.2

Learn to add the following statement to questions asking "*are they perpendicular?*". $\vec{c} \cdot \vec{d} = 0$ therefore $\cos x = 0$, therefore $x = 90°$. Lines are perpendicular. Of course, when lines are not perpendicular replace all $=$ with $\neq$.

The dot product of two vectors $\vec{c} \cdot \vec{d}$ can be used to find the angle between them. Let

$$\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \qquad \vec{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$$\vec{c} \cdot \vec{d} = |\vec{c}||\vec{d}|\cos\theta$$
$$\vec{c} \cdot \vec{d} = c_1 d_1 + c_2 d_2 + c_3 d_3$$

Finding the angle between two lines.
(Often are these two vectors perpendicular)

Find the angle between $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 1 \\ 3 \end{pmatrix}$.

1.	Find $\vec{c} \cdot \vec{d}$ in terms of components	$\vec{c} \cdot \vec{d} = 2 \times 8 + 3 \times 1 + (-1) \times 3 = 16$
2.	Find $\vec{c} \cdot \vec{d}$ in terms of magnitudes	$\vec{c} \cdot \vec{d} = \sqrt{2^2 + 3^2 + (-1)^2} \times \sqrt{8^2 + 1^2 + 3^2} \times \cos\theta = \sqrt{14}\sqrt{74}\cos\theta$
3.	Equate and solve for θ	$16 = \sqrt{14}\sqrt{74}\cos\theta$ $\Rightarrow \cos\theta = \dfrac{16}{\sqrt{14}\sqrt{74}}$ $\Rightarrow \theta = 60.2°$

Note: when $\theta = 90°$ (perpendicular vectors), $\cos(90°) = 0 \Rightarrow \vec{c} \cdot \vec{d} = 0$

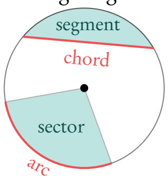
TRIGONOMETRY AND CIRCULAR FUNCTIONS

Table of contents & cheatsheet

5.1. Basic trigonometry 38

radians $= \frac{\pi}{180°} \times$ degrees   degrees $= \frac{180°}{\pi} \times$ radians

Before each question make sure calculator is in correct setting: degrees or radians?

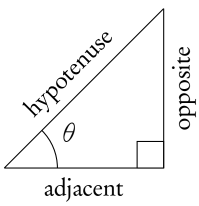


Area of a sector $= \frac{1}{2} r^2 \cdot \theta$

Arc length $= r \cdot \theta$

θ in radians, $r =$ radius.

Right-angle triangle (triangle with 90° angle)

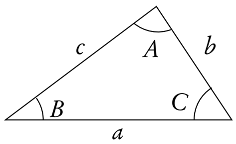


$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ SOH

$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ CAH

$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$ TOA

Non-right angle triangles



Sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Use this rule when you know: 2 angles and a side (not between the angles) or 2 sides and an angle (not between the sides).

Cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$

Use this rule when you know: 3 sides or 2 sides and the angle between them.

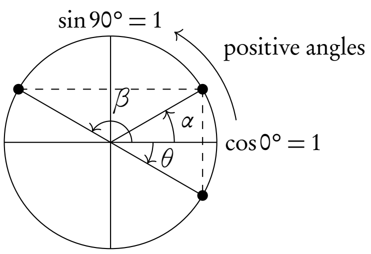
Area of a triangle: Area $= \frac{1}{2} ab\sin C$

Use this rule when you know: 3 sides or 2 sides and the angle between them.

Three-figure bearings

Direction given as an angle of a full circle. North is 000 and the angle is expressed in the clockwise direction from North. So East is 090, South is 180 and West 270.

5.2. Circular functions 43



deg	0°	30°	45°	60°	90°	120°	135°	150°	180°
rad	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	$\frac{2}{3}\pi$	$\frac{3}{4}\pi$	$\frac{5}{6}\pi$	π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

Trigonometric function $y = a\sin(bx + c) + d$

Amplitude: a

Period: $\frac{360°}{b}$ or $\frac{2\pi}{b}$

Horizontal shift: c

Vertical shift: d

Trigonometric identities

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sin^2\theta + \cos^2\theta = 1$

$2\sin\theta\cos\theta = \sin 2\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

ACADEMY

5.1 Basic trigonometry

This section offers an overview of some basic trigonometry rules and values that will recur often. It is worthwhile to know these by heart; but it is much better to understand how to obtain these values. Like converting between Celsius and Fahrenheit; you can remember some values that correspond to each other but if you understand how to obtain them, you will be able to convert any temperature.

5.1.1 Converting between radians and degrees

$$\text{radians} = \frac{\pi}{180°} \times \text{degrees}$$

$$\text{degrees} = \frac{180°}{\pi} \times \text{radians}$$

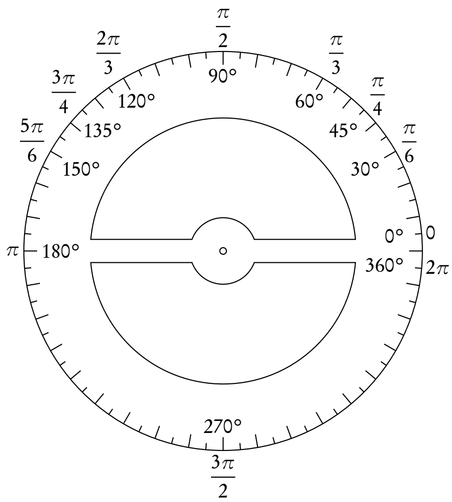


Table 5.1: Common radians/degrees conversions

Degrees	0°	30°	45°	60°	90°	120°	135°	180°	270°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π

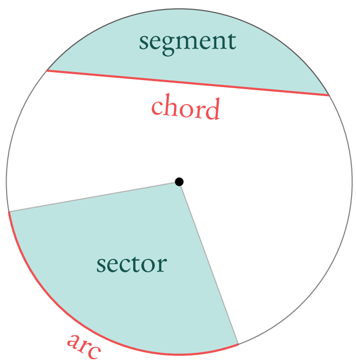
5.1.2 Circle formulas

DB 3.1

Area of a sector $= \frac{1}{2} r^2 \cdot \theta$

Arc length $= r \cdot \theta$

θ in radians, $r =$ radius.



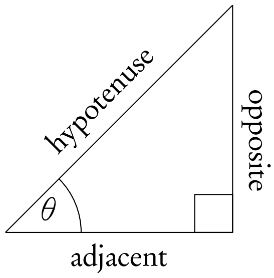
5.1.3 Right-angle triangles

$$a^2 = b^2 + c^2 \qquad \text{Pythagoras}$$

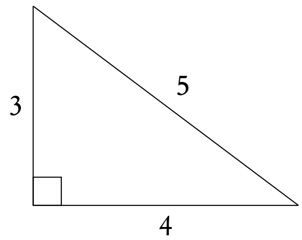
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \text{SOH}$$

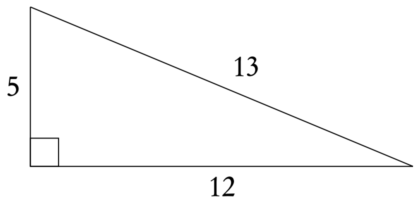
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \text{CAH}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \text{TOA}$$

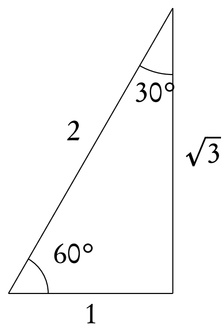


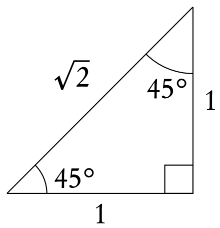
Two important triangles to memorize:





The IB loves asking questions about these special triangles which have whole numbers for all the sides of the right triangles.



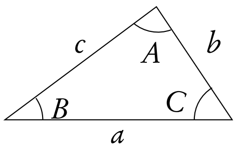


Note: these triangles can help you in finding the sin, cos and tan of the angles that you should memorize, shown in table 5.2 at page 44. Use SOH, CAH, TOA to find the values.

Read the question, does it specify if you are looking for an acute (less than 90°) or obtuse (more than 90°) angle. If not there may be 2 solutions. Exam hint: Use sketches when working with worded questions!

DB 3.6

5.1.4 Non-right angle triangles

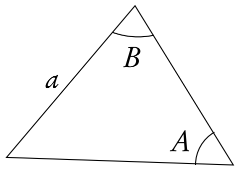


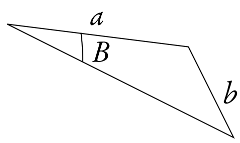
To find any missing angles or side lengths in non-right angle triangles, use the *cosine* and *sine* rule. Remember that the angles in the triangle add up to 180°!

Sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Use this rule when you know:

2 angles and a side (not between the angles) — or — 2 sides and an angle (not between the sides)

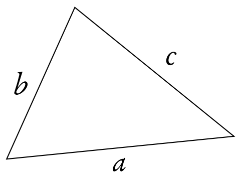


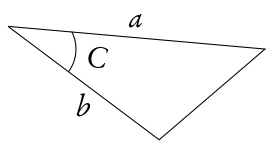


Cosine rule: $c^2 = a^2 + b^2 - 2\,a\,b\cos C$

Use this rule when you know:

3 sides — or — 2 sides and the angle between them

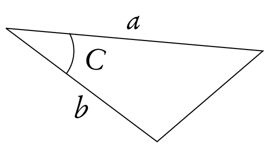


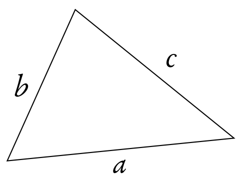


Area of a triangle: Area $= \dfrac{1}{2} a\,b\sin C$

Use this rule when you know:

2 sides and the angle between them — or — 3 sides
first you need to use cosine rule
to find an angle





Example.

$\triangle ABC : A = 40°, B = 73°, a = 27\,\text{cm}.$

Find $\angle C$.

$$\angle C = 180° - 40° - 73° = 67°$$

Find b.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{27}{\sin 40°} = \frac{b}{\sin 73°}$$
$$b = \frac{27}{\sin 40°} \cdot \sin 73° = 40.169 \approx 40.2\,\text{cm}$$

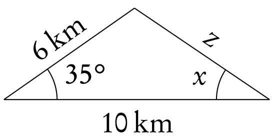
Find c.

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$c = \frac{27}{\sin 40°} \times \sin 67° = 38.7\,\text{cm}$$

Find the area.

$$\text{Area} = \frac{1}{2} \cdot 27 \cdot 40 \cdot 2 \cdot \sin 67°$$
$$= 499.59 \approx 500\,\text{cm}^2$$

Example.



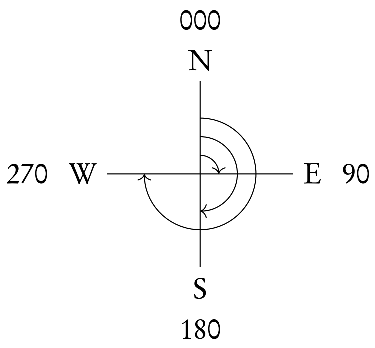
Find z.

$$z^2 = 6^2 + 10^2 - 2 \cdot 6 \cdot 10 \cdot \cos 35°$$
$$z^2 = 37.70$$
$$z = 6.14\,\text{km}$$

Find $\angle x$.

$$\frac{6}{\sin x} = \frac{6.14}{\sin 35°}$$
$$\sin x = 0.56$$
$$x = \sin^{-1}(0.56) = 55.91°$$

5.1.5 Three-figure bearings

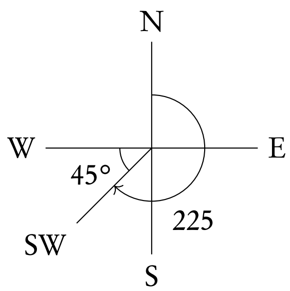


Three-figure bearings can be used to indicate compass directions on maps. They will be given as an angle of a full circle, so between 000 and 360. North is always marked as 000. Any direction from there can be expressed as the angle in the clockwise direction from North.

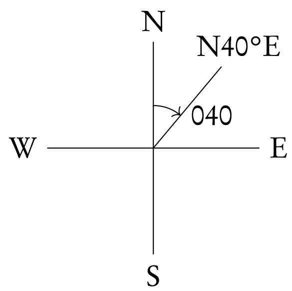
In questions on three-figure bearings, you are often confronted with quite a lot of text, so it is a good idea to first make a drawing. You may also need to create a right angle triangle and use your basic trigonometry.

Example.

SW: 45° between South and West = 225



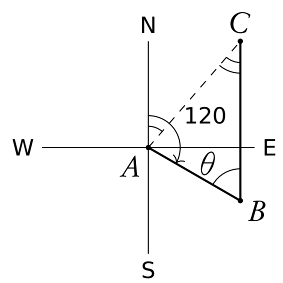
N40°E: 40° East of North = 040



A ship left port A and sailed $20\,\mathrm{km}$ in the direction 120.

It then sailed north for $30\,\mathrm{km}$ to reach point C. How far from the port is the ship?

1. Draw a sketch



2. Find an internal angle of the triangle.

$\theta = 180° - 120° = 60° = C$

Similar angles between two parallel lines

3. Use cosine or sin rule.

(here cosine)

$$AC^2 = AB^2 + BC^2 - 2 \cdot AB \cdot BC \cdot \cos\theta$$

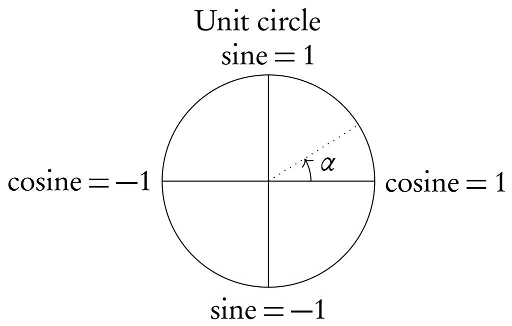
$$AC^2 = 20^2 + 30^2 - 2 \cdot 20 \cdot 30 \cdot \cos 60°$$

$$AC^2 = 400 + 900 - 2 \cdot 20 \cdot 30 \cdot \frac{1}{2}$$

$$AC = \sqrt{400 + 900 - 600} = \sqrt{700}$$

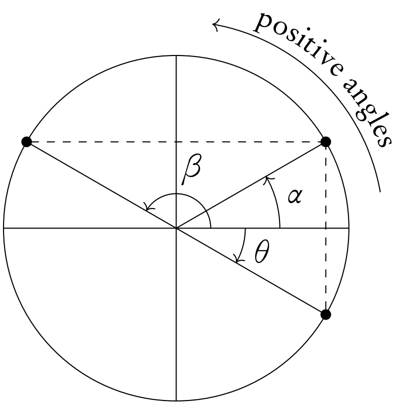
5.2 Circular functions

5.2.1 Unit circle



The unit circle is a circle with a radius of 1 drawn from the origin of a set of axes. The y-axis corresponds to *sine* and the x-axis to *cosine*; so at the coordinate $(0,1)$ it can be said that cosine $=0$ and sine $=1$, just like in the $\sin x$ and $\cos x$ graphs when plotted.

The unit circle is particularly useful to find all the solutions to a trigonometric equation within a certain domain. As you can see from their graphs, functions with $\sin x$, $\cos x$ or $\tan x$ repeat themselves every given period; this is why they are also called *circular functions*. As a result, for each y-value there is an infinite amount of x-values that could give you this output. This is why questions will give you a set domain that limits the range of x-values you should consider in your calculations or represent on your sketch (e.g. $0° \leq x \leq 360°$).

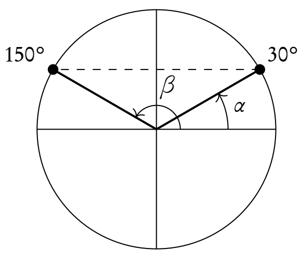


Relations between sin, cos and tan:

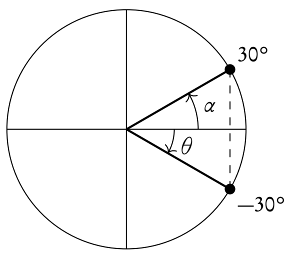
- α and β have the same sine
- α and θ have the same cosine
- β and θ have the same tangent

Example.

$\sin 30° = \sin 150°$



$\cos 30° = \cos 330°$



$\tan 150° = \tan 330°$

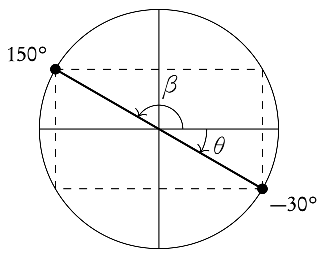
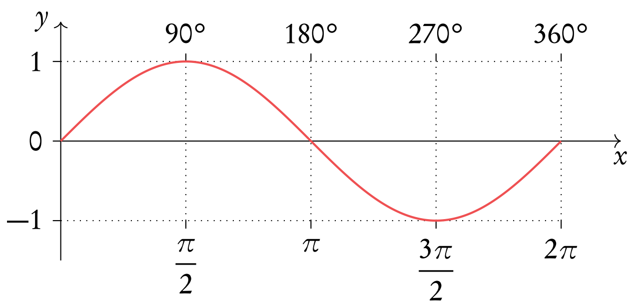


Table 5.2: Angles to memorize

deg	0°	30°	45°	60°	90°	120°	135°	150°	180°
rad	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	$\frac{2}{3}\pi$	$\frac{3}{4}\pi$	$\frac{5}{6}\pi$	π
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

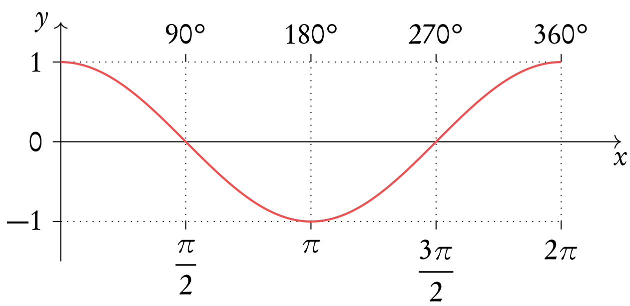
5.2.2 Graphs: trigonometric functions

sin x



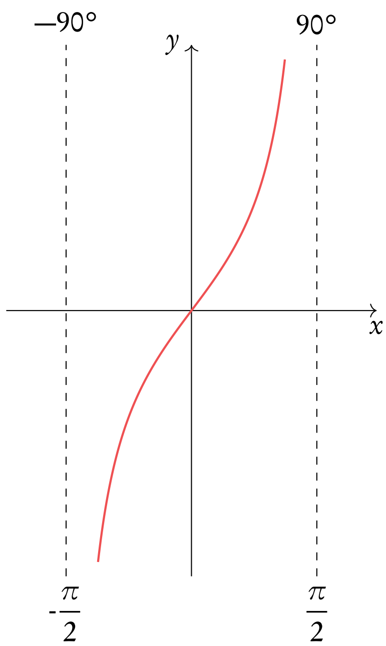
Domain: $x \in \mathbb{R}$
Amplitude: $-1 \leq y \leq 1$
Period: $2n\pi$, $n \cdot 360°$, with $n \in \mathbb{Z}$

cos x



Domain: $x \in \mathbb{R}$
Amplitude: $-1 \leq y \leq 1$
Period: $2n\pi$, $n \cdot 360°$, with $n \in \mathbb{Z}$

tan x



Domain: $x \in \mathbb{R}$, $x \neq \frac{\pi}{2} + k\pi$, with $k \in \mathbb{Z}$
Amplitude: $-\infty < y < \infty$
Period: $n\pi$, $n \cdot 180°$, with $n \in \mathbb{Z}$

5.2.3 Transformations

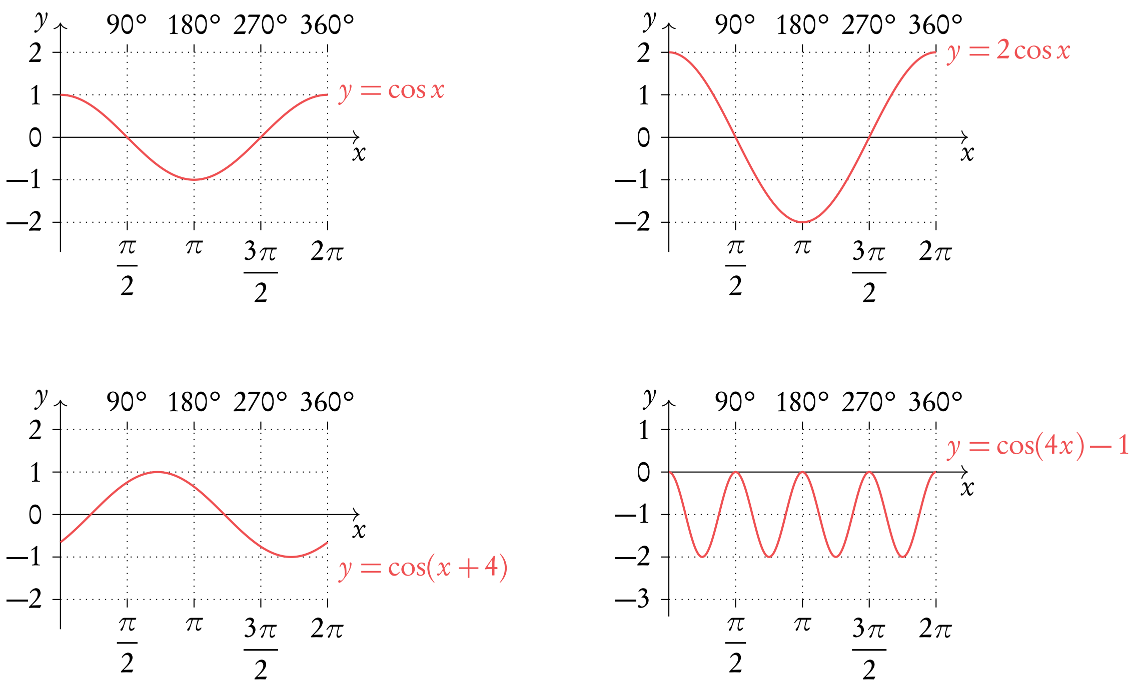
Besides the transformations in the functions chapter, trigonometric functions have some transformations with their own particular names. For a trigonometric function, the vertical stretch on a graph is determined by its amplitude, the horizontal stretch by its period and an upward/downward shift by its axis of oscillation.

A trigonometric function, given by $y = a\sin(bx + c) + d$, has:

- an amplitude a;
- a period of $\frac{360°}{b}$ or $\frac{2\pi}{b}$;
- a horizontal shift of $+c$ *to the left*, in degrees or radians;
- vertical shift of $+d$ *upwards*, oscillates around d.

Example.

Transformations of $y = \cos x$.



5.2.4 Identities and equations

DB 3.2 & 3.3

In order to solve trigonometric equations, you will sometimes need to use identities. Identities allow you to rewrite your equation in a way that will make it easier to solve algebraically.

Trigonometric identity	Popular rearrangement
$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	$\cos\theta = \dfrac{\sin\theta}{\tan\theta}$ $\sin\theta = \cos\theta \times \tan\theta$
$\sin^2\theta + \cos^2\theta = 1$	$\sin^2\theta = 1 - \cos^2\theta$ $\cos^2\theta = 1 - \sin^2\theta$
$2\sin\theta\cos\theta = \sin 2\theta$	
$\cos 2\theta = \cos^2\theta - \sin^2\theta$	$\cos 2\theta = 1 - 2\sin^2\theta$ $\cos 2\theta = 2\cos^2\theta - 1$

Solving equations with trigonometric identities

Solve $2\cos^2 x + \sin x = 1$, $0° \le x \le 360°$.

1.	Identify which identity from the databook to use. Note you are always aiming to get an equation with just, sin, cos or tan.	Here we could use either $\sin^2\theta + \cos^2\theta = 1$ or $\cos^2\theta - \sin^2\theta = \cos 2\theta$. We will use the first so that we get an equation with just sin.
2.	Rearrange identity and substitute into equation.	$\cos^2\theta = 1 - \sin^2\theta$ $2(1 - \sin^2 x) + \sin x = 1$ $2 - 2\sin^2 x + \sin x = 1$ $-2\sin^2 x + \sin x + 1 = 0$
3.	Solve for x. Giving answers within the stated range. Recognise that here the eqauation looks like a quadratic equation.	Substitue u for $\sin x$: $-2u^2 + u + 1 = 0$ $(-2u - 1)(u - 1) = 0$ $u = \sin x \Rightarrow 1 \qquad x \Rightarrow 90°$ $u = \sin x \Rightarrow -0.5 \qquad x \Rightarrow 210°$ or $330°$

DIFFERENTIATION

6

Table of contents & cheatsheet

Definitions

Differentiation is a way to find the gradient of a function at any point, written as $f'(x)$, y' and $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Tangent line to a point on a curve is a linear line with the same gradient as that point on the curve.

6.2. Polynomials 48

Product $y = uv$, then: $y' = uv' + u'v$

Quotient $y = \frac{u}{v}$, then: $y' = \frac{vu' - uv'}{v^2}$

Chain $y = g(u)$ where $u = f(x)$, then:
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

6.3. Tangent and normal 51

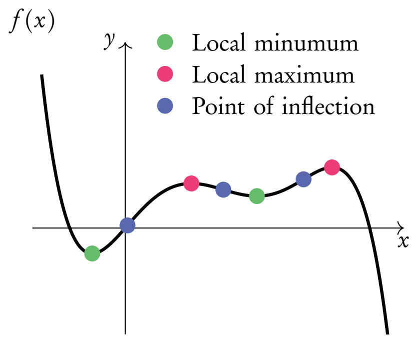
Tangent line with the same gradient as a point on a curve.

Normal perpendicular to the tangent $m = \frac{-1}{\text{slope of tangent}}$

Both are linear lines with general formula: $y = mx + c$.

1. Use derivative to find gradient of the tangent. For normal then do $-\frac{1}{\text{slope of tangent}}$.
2. Input the x-value of the point into $f(x)$ to find y.
3. Input y, m and the x-value into $y = mx + c$ to find c.

6.4. Turning points 52



	$f'(x)$	$f''(x)$
Local minimum	0	+
Local maximum	0	−
Points of inflection		0

6.5. Sketching graphs 56

Gather information before sketching:

Intercepts x-intercept: $f(x) = 0$
y-intercept: $f(0)$

Turning points minima: $f'(x) = 0$ and $f''(x) < 0$
maxima: $f'(x) = 0$ and $f''(x) > 0$
point of inflection: $f''(x) = 0$

Asymptotes vertical: x-value when the function divides by 0
horizontal: y-value when $x \to \infty$

Plug the found x-values into $f(x)$ to determine the y-values.

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Kinematics

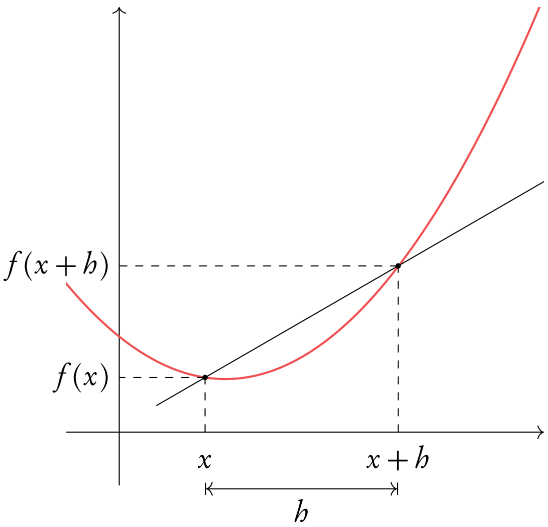
Derivative represents the rate of change, integration the reverse.

$\frac{\mathrm{d}s}{\mathrm{d}t} = v$ $\frac{\mathrm{d}v}{\mathrm{d}t} = a$

s displacement, v velocity, a acceleration

6.1 Derivation from first principles

As the derivative at a point is the gradient, differentiation can be compared to finding gradients of lines: $m = \frac{y_2 - y_1}{x_2 - x_1}$.



Using the graph

$$x_1 = x \qquad x_2 = x + h$$
$$y_1 = f(x) \qquad y_2 = f(x+h)$$

Plugging into the equation of the gradient of a line

$$m = \frac{f(x+h) - f(x)}{x + h - x}$$

Taking the limit of h going to zero, such that the distance between the points becomes very small, one can approximate the gradient at a point of any funtion:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

6.2 Polynomials

As you have learnt in the section on functions, a straight line graph has a gradient. This gradient describes the rate at which the graph is changing and thanks to it we can tell how steep the line will be. In fact gradients can be found for any function - the special thing about linear functions is that their gradient is always the same (given by m in $y = mx + c$). For polynomial functions the gradient is always changing. This is where calculus comes in handy; we can use differentiation to derive a function using which we can find the gradient for any value of x.

Using the following steps, you can find the derivative function ($f'(x)$) for any polynomial function ($f(x)$).

Polynomial a mathematical expression or function that contains several terms often raised to different powers

e.g. $y = 3x^2$, $y = 121x^5 + 7x^3 + x$ or $y = 4x^{\frac{2}{3}} + 2x^{\frac{1}{3}}$

Principles $y = f(x) = ax^n \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = nax^{n-1}$.

The (original) function is described by y or $f(x)$, the derivative (gradient) function is described by $\frac{\mathrm{d}y}{\mathrm{d}x}$ or $f'(x)$.

Derivative of a constant (number) 0

e.g. For $f(x) = 5$, $f'(x) = 0$

Derivative of a sum sum of derivatives.

If a function you are looking to differentiate is made up of several summed parts, find the derivatives for each part separately and then add them together again.

e.g. $f(x) = ax^n$ and $g(x) = bx^m$

$$f'(x) + g'(x) = nax^{n-1} + mbx^{m-1}$$

6.2.1 Rules

With more complicated functions, in which several functions are being multiplied or divided by one another (rather than just added or substracted), you will need to use the product or quotient rules. DB 6.2

Product rule

When functions are multiplied: $y = uv$

then: $y' = uv' + u'v$

which is the same as $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$.

e.g. $y = x^2 \cos x$, then $y' = x^2(\cos x)' + (x^2)' \cos x = -x^2 \sin x + 2x \cos x$

Quotient rule

When functions are divided: $y = \frac{u}{v}$

then: $y' = \frac{vu' - uv'}{v^2}$

which is the same as $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$.

e.g. $y = \frac{x^2}{\cos x}$, then $y' = \frac{(x^2)'\cos x - x^2(\cos x)'}{(\cos x)^2} = \frac{2x\cos x + x^2 \sin x}{\cos^2 x}$

Chain rule

Function inside another function: $y = g(u)$ where $u = f(x)$

then: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$.

Differentiating with the chain rule.

Let $y = (\cos x)^2$, determine the derivative y'

1.	What is the outside function? What is the inside function?	Inside function: $u = \cos x$ Outside function: $y = u^2$
2.	Find u' and y'	$u' = \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x;\quad y' = \frac{\mathrm{d}y}{\mathrm{d}u} = 2u$
3.	Fill in formula	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$ $= 2u(-\sin x)$ $= -2\sin x \cos x$

6.3 Tangent and normal equation

Tangent a straight line that touches a curve at one single point. At that point, the gradient of the curve is equal to the gradient of the tangent.

Normal a straight line that is perpendicular to the tangent line:

$$\text{slope of normal} = \frac{-1}{\text{slope of tangent}}$$

For any questions with tangent and/or normal lines, use the steps described in the following example.

Finding the linear function of the tangent.

Let $f(x) = x^3$. Find the equation of the tangent at $x = 2$

1.	Find the derivative and fill in value of x to determine slope of tangent	$f'(x) = 3x^2$ $f'(2) = 3 \cdot 2^2 = 12$
2.	Determine the y value	$f(x) = 2^3 = 8$
3.	Plug the slope m and the y value in $y = mx + c$	$8 = 12x + c$
4.	Fill in the value for x to find c	$8 = 12(2) + c \Rightarrow c = -16$ eq. of tangent: $y = 12x - 16$

Steps 1, 2 and 4 are identical for the equation of the tangent and normal

Finding the linear function of the normal.

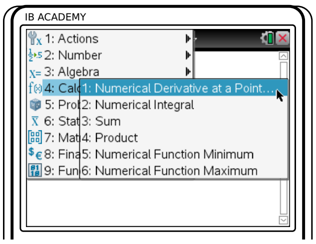
Let $f(x) = x^3$. Find the equation of the normal at $x = 2$

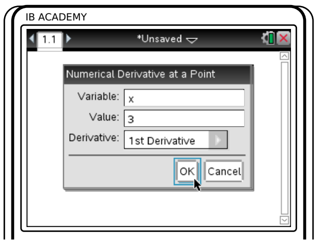
1.		$f'(2) = 12$
2.		$f(x) = 8$
3.	Determine the slope of the normal $m = \dfrac{-1}{\text{slope tangent}}$ and plug it and the y-value into $y = mx + c$	$m = \dfrac{-1}{12}$ $8 = -\dfrac{1}{12}x + c$
4.	Fill in the value for x to find c	$8 = -\dfrac{1}{12}(2) + c \Rightarrow c = \dfrac{49}{6}$ eq. of normal: $y = -\dfrac{1}{12}x + \dfrac{49}{6}$

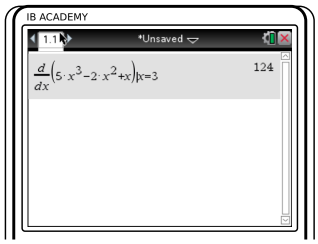
Steps 1, 2 and 4 are identical for the equation of the tangent and normal

To find the gradient of a function for any value of x.

$f(x) = 5x^3 - 2x^2 + x$. Find the gradient of $f(x)$ at $x = 3$.







Press menu
`4: Calculus`
`1: Numerical Derivative at a Point`

Enter the variable used in your function (x) and the value of x that you want to find. Keep the settings on `1st Derivative`
Press OK

Type in your function
press enter

In this case, $f'(3) = 124$

6.4 Turning points

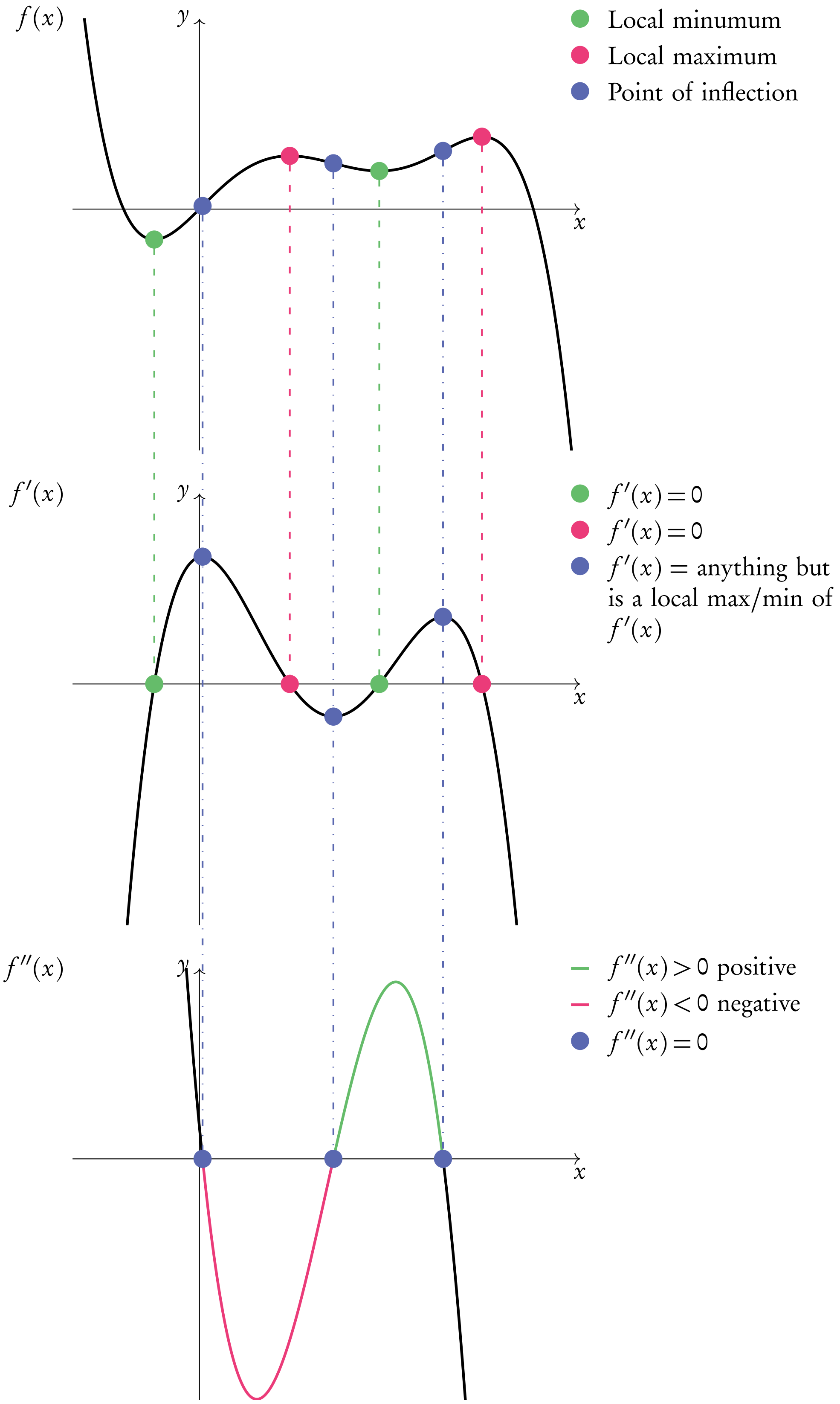
There are three types of turning points:

1. Local maxima
2. Local minima
3. Points of inflection

We know that when $f'(x) = 0$ there will be a maximum or a minimum. Whether it is a maximum or minimum should be evident from looking at the graph of the original function. If a graph is not available, we can find out by plugging in a slightly smaller and slightly larger value than the point in question into $f'(x)$. If the smaller value is negative and the larger value positive then it is a local minimum. If the smaller value is positive and the larger value negative then it is a local minimum.

If you take the derivative of a derivative function (one you have already derived) you get the *second derivate*. In mathematical notation, the second derivative is written as y'', $f''(x)$ or $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. We can use this to determine whether a point on a graph is a maximum, a minimum or a point of inflection as demonstrated in the following Figure 6.1.

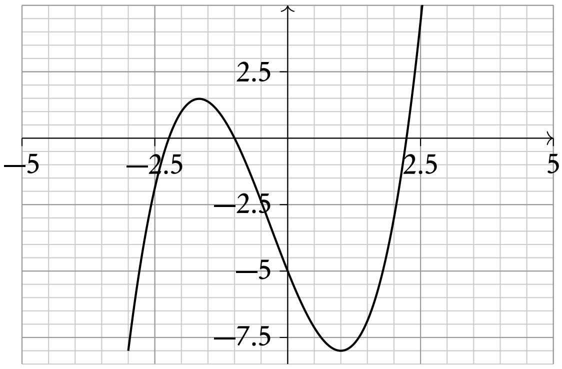
Figure 6.1: Graph that shows a local maximum, a local minimum and points of inflection



Notice how the points of inflection of $f(x)$ are minima and maxima in $f'(x)$ and thus equal 0 in $f''(x)$

Finding turning points.

The function $f(x) = x^3 + x^2 - 5x - 5$ is shown. Use the first and second derivative to find turning points: the minima, maxima and points of inflection (POI).



1. Find the first and second derivative.

$$f'(x) = 3x^2 + 2x - 5$$
$$f''(x) = 6x + 2$$

2. Find x_{min} and x_{max} by setting $f'(x) = 0$.

$$3x^2 + 2x - 5 = 0$$

GDC yields: $x = 1$ or $x = -\frac{5}{3}$

3. Find y-coordinates by inserting the x-value(s) into the original $f(x)$.

$f(1) = (1)^3 + (1)^2 - 5(1) - 5 = -8$,
so x_{min} at $(1, -8)$.

$$f\left(-\frac{5}{3}\right) = \left(-\frac{5}{3}\right)^3 + \left(-\frac{5}{3}\right)^2 - 5\left(-\frac{5}{3}\right) - 5 = 1.48 (3 \text{ s.f.}),$$

so x_{max} at $\left(-\frac{5}{3}, 1.48\right)$.

4. Find POI by setting $f''(x) = 0$

$$6x + 2 = 0$$

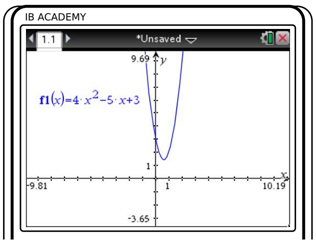
5. Then enter values of x into original function to find coordinates

$$f\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^3 + \left(-\frac{1}{3}\right)^2 - 5\left(-\frac{1}{3}\right) - 5$$
$$y = -3.26 \text{ (3 s.f.)}$$

so POI at $\left(-\frac{1}{3}, -3.26\right)$

To find turning points (local maximum/minimum) of a function

Find the coordinates of the local minimum for $f(x) = 4x^2 - 5x + 3$

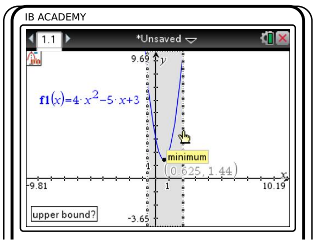


Press menu

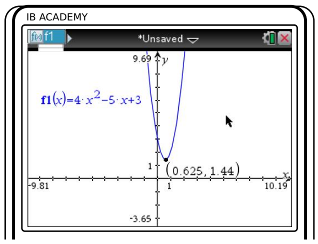
`6: Analyze graph`

or `2: Minimum`

or `3: Maximum`



Use the cursor to set the bounds (the min/max must be between the bounds)



So the coordinates of the minimum for $f(x)$ are $(0.625, 1.44)$

6.5 Sketching graphs

When sketching a graph, you will need the following information:

1. Intercepts,
2. Turning points (maximums, minimums and inflection points) and
3. Asymptotes

Sketching a function.

Sketch the function $f(x) = \dfrac{x^2}{x^2-16}$

1. Note down all information:

1. Intercepts:
   - y-intercept: $f(0)$
   - x-intercept: $f(x) = 0$

1. y-intercept when $x = 0$:

   $f(0) = \dfrac{0^2}{0^2-16} = 0 \quad (0,0)$

   $f(x) = \dfrac{x^2}{x^2-16} = 0 \quad x = 0 \quad$ (same)

   This is the only x-intercept.

2. Turning points:
   - min/max: $f'(x) = 0$
   - inflection: $f''(x) = 0$

2. Turning point: $f'(x) = \dfrac{-32x}{x^2-16^2}$,

   $x = 0$ $(0,0)$ (Found with quotient rule).

   $f' = 0$ when $x = 0$.

3. Asymptotes:
   - vertical: denominator = 0, $x = -b$, for $\log(x+b)$
   - horizontal: $\lim\limits_{x\to\infty \text{ or } x\to-\infty}$, $y = c$, for $a^x + c$

3. Vertical asymptotes when $x^2 - 16 = 0$, so $x = 4$ and $x = -4$.

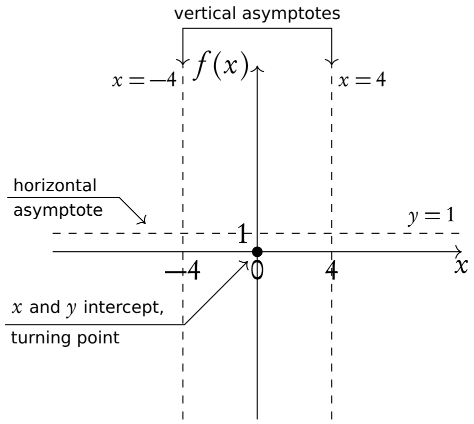
   Horizontal asymptote:

   $\lim\limits_{x\to\infty} f(x) = \dfrac{x^2}{x^2} = 1$, so $y = 1$

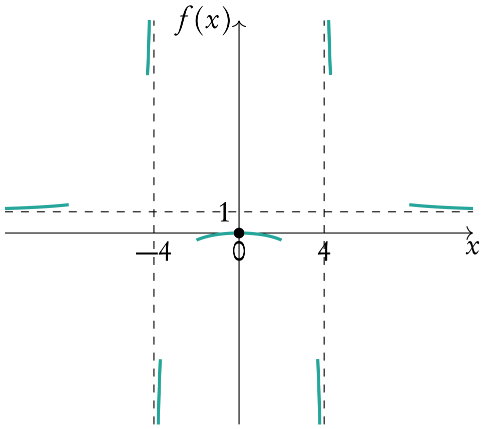
To find the y-coordinate, input the x-value into the original $f(x)$.

2. Mark out information on axis

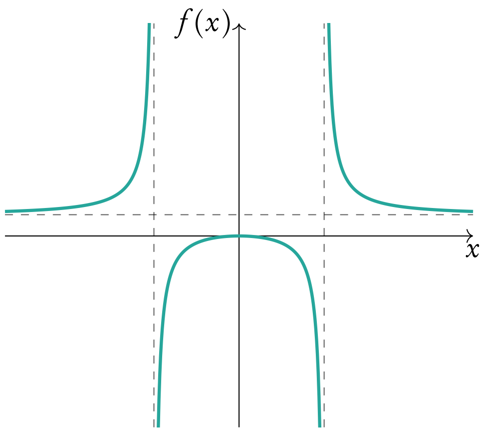
Clearly indicate them to guarantee marks



3. Think about where your lines are coming from



4. Join the dots



6.6 Applications

6.6.1 Kinematics

Kinematics deals with the movement of bodies over time. When you are given one function to calculate displacement, velocity or acceleration you can use differentiation or integration to determine the functions for the other two.

$$\begin{array}{ccc} & \text{Displacement, } s & \\ \dfrac{ds}{dt}\ \Big\downarrow & & \Big\uparrow\ \displaystyle\int v\,dt \\ & \text{Velocity, } v = \dfrac{ds}{dt} & \\ \dfrac{dv}{dt}\ \Big\downarrow & & \Big\uparrow\ \displaystyle\int a\,dt \\ & \text{Acceleration, } a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2} & \end{array}$$

The derivative represents the rate of change, i.e. the gradient of a graph. So, velocity is the rate of change in displacement and acceleration is the rate of change in velocity.

Answering kinematics questions.

A diver jumps from a platform at time $t = 0$ seconds. The distance of the diver above water level at time t is given by $s(t) = -4.9t^2 + 4.9t + 10$, where s is in metres. Find when velocity equals zero. Hence find the maximum height of the diver.

1.	Find an equation for velocity by differentiating equation for distance	$v(t) = -9.8t + 4.9$
2.	Solve for $v(t) = 0$	$-9.8t + 4.9 = 0, \quad t = 0.5$
3.	Put value into equation for distance to find height above water	$s(0.5) = -4.9(0.5)^2 + 4.9(0.5) + 10 = 11.225\,\text{m}$

6.6.2 Optimization

We can use differentiation to find minimum and maximum areas/volumes of various shapes. Often the key skill with these questions is to find an expression using simple geometric formulas and rearranging in order to differentiate.

Finding the min/max area or volume

The sum of height and base of a triangle is $40\,\text{cm}$. Find an expression for its area in terms of x, its base length. Hence find its maximum area.

1.	Find expressions for relevant dimensions of the shape	length of the base $(b) = x$ height + base $= 40$ so $h + x = 40$ area of triangle $A = \frac{1}{2}xh$
2.	Reduce the number of variables by solving the simultaneous equations	Since $h = 40 - x$, substitute h into A: $A = \frac{1}{2}x(40 - x) = -\frac{1}{2}x^2 + 20x$
3.	Differentiate	$f'(x) = -x + 20$
4.	Find x when $f'(x) = 0$	$-x + 20 = 0 \Rightarrow x = 20$
5.	Plug x value in $f(x)$	$-\frac{1}{2}20^2 + 20(20) = -200 + 400 = 200\,\text{cm}^2$

If an expression is given in the problem, skip to step 2 (e.g. cost/profit problems).

INTEGRATION

7

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7.1. Indefinite integral 62

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

Integration with an internal function

$$\int f(ax+b)\,dx$$

Integrate normally and multiply by $\dfrac{1}{\text{coefficient of } x}$

Integration by substitution

$$\int f\left(g(x)\right) \cdot g'(x)\,dx$$

7.2. Definite integral 64

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad \text{where} \qquad F = \int f(x)\,dx$$

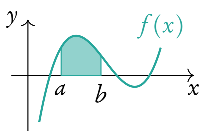
Be careful, the order you substitute a and b into the indefinite integral is relevant for your answer:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Area between a curve and the x-axis

By determining a definite integral for a function, you can find the area beneath the curve that is between the two x-values indicated as its limits.

$$A_{\text{curve}} = \int_a^b f(x)\,dx$$

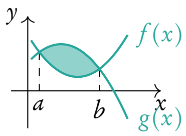


Note: the area below the x-axis gives a negative value for its area. You must take that value as a positive value to determine the area between a curve and the x-axis. Sketching the graph will show what part of the function lies below the x-axis.

Area between two curves

Using definite integrals you can also find the areas enclosed between curves.

$$A_{\text{between}} = \int_a^b \left(g(x) - f(x)\right)dx$$



With $g(x)$ as the "top" function (furthest from the x-axis). For the area between curves, it does not matter what is above/below the x-axis.

Volume of revolution

$$V = \pi \int_a^b y^2\,dx = \int_a^b \pi y^2\,dx$$

Besides finding areas under and between curves, integration can also be used to calculate the volume of the solid that a curve would make if it were rotated 360° around its axis - this is called the volume of revolution.

ACADEMY

7.1 Indefinite integral and boundary condition

Integration is essentially the opposite of derivation. The following equation shows how to integrate a function:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$$

As you can see, every time you integrate the power on your variable will increase by 1 (this is opposite of what happens when you derive, then it always decreases). Whenever you integrate you also **always add** $+C$ to this function. This accounts for any constant that may have been lost while deriving. As you may have noticed, whenever you do derivation any constants that were in the original function, $f(x)$, become 0 in the derivative function, $f'(x)$. In order to determine the value of C, you need to fill in a point that lies on the curve to set up an equation in which you can solve for C. (Note: this is the same thing you need to do when finding the y-intercept, C, for a linear function – see Functions: Linear functions).

Standard integration.

Let $f'(x) = 12x^2 - 2$
Given that $f(-1) = 1$, find $f(x)$.

1.	Separate summed parts (optional)	$\int 12x^2 - 2\,dx = \int 12x^2\,dx + \int -2\,dx$
2.	Integrate	$f(x) = \int 12x^2\,dx + \int -2\,dx = \frac{12}{3}x^3 - 2x + C$
3.	Fill in values of x and $f(x)$ to find C	Since $f(-1) = 1$, $4(-1)^3 - 2(-1) + C = 1$ $C = 3$ So: $f(x) = 4x^3 - 2x + 3$

7.1.1 Integration with an internal function

$$\int f(ax+b)\,\mathrm{d}x \qquad \text{integrate normally and multiply by } \frac{1}{\text{coefficient of } x}$$

Example.

Find the following integrals:

$$\int e^{3x-4}\,\mathrm{d}x$$

Coefficient of $x=3$, so

$\int e^{3x-4}\,\mathrm{d}x = \frac{1}{3}e^{3x-4}+C$

$$\int \cos(5x-2)\,\mathrm{d}x$$

Coefficient of $x=5$, so

$\int \cos(5x-2)\,\mathrm{d}x = \frac{1}{5}\sin(5x-2)+C$

7.1.2 Integration by substitution

$$\int f\left(g(x)\right)\cdot g'(x)\,\mathrm{d}x$$

Integration by substitution: usually these questions will be the most complicated-looking integrals you will have to solve. So if an integration question looks complicated, try to look for a function and its derivative inside the function you are looking to integrate; it is likely to be a question where you have to use the substitution method! Study the example to see how it's done.

Integrate by substitution

Find $\int 3x^2 e^{x^3}\,\mathrm{d}x$

1.	Identify the inside function u, this is the function whose derivative is also inside $f(x)$.	$g(x)=u=x^3$
2.	Find the derivative $u'=\frac{\mathrm{d}u}{\mathrm{d}x}$	$\frac{\mathrm{d}u}{\mathrm{d}x}=3x^2$
3.	Substitute u and $\frac{\mathrm{d}u}{\mathrm{d}x}$ into the integral (this way $\mathrm{d}x$ cancels out)	$\int e^u \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int e^u\,\mathrm{d}u = e^u+C$
4.	Substitute u back to get a function with x	$\int e^u + C = e^{x^3}+C$

7.2 Definite integral

If there are limit values indicated on your integral, you are looking to find a definite integral. This means that these values will be used to find a numeric answer rather than a function.

This is done in the following way, where the values for a and b are substituted as x-values into your indefinite integral:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) \qquad \text{where} \qquad F = \int f(x)\,\mathrm{d}x$$

Be careful, the order you substitute a and b into the indefinite integral is relevant for your answer:

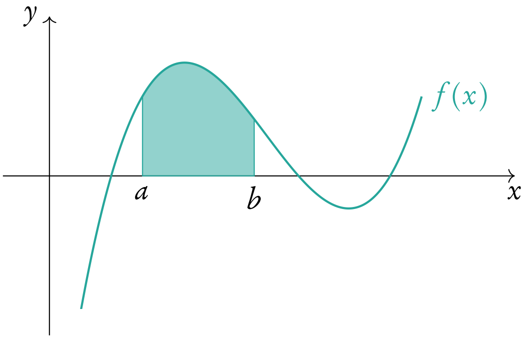
$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$

	Solving definite integrals.	
	Find $\int_3^7 12x^2 - 2\,\mathrm{d}x$, knowing that $F(x) = 4x^3 - 2x$	
1.	Find the indefinite integral (without $+C$)	$\int_3^7 12x^2 - 2\,\mathrm{d}x = \left[4x^3 - 2x\right]_3^7$
2.	Fill in: $F(b) - F(a)$ (integral $x = b$) $-$ (integral $x = a$)	$= \left[4(7)^3 - 2(7)\right] - \left[4(3)^3 - 2(3)\right]$ $= 1256$

ACADEMY

7.2.1 Area

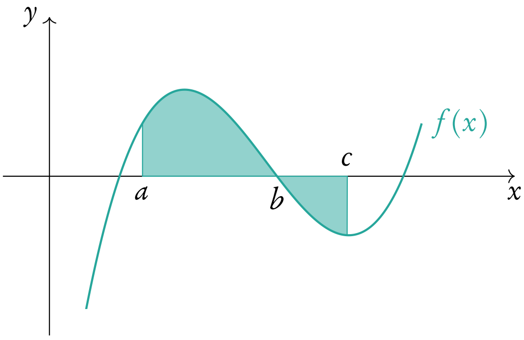
Area between a curve and the x-axis



DB 6.5

By determining a definite integral for a function, you can find the area beneath the curve that is between the two x-values indicated as its limits.

$$A_{\text{curve}} = \int_a^b f(x)\,\mathrm{d}x$$



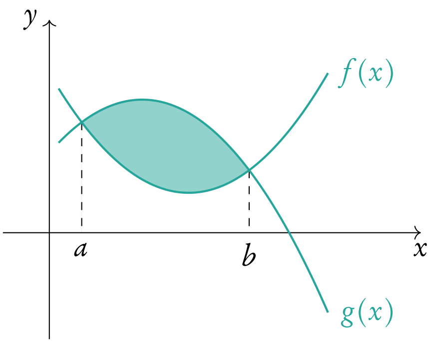
Note: the area below the x-axis gives a negative value for its area. You must take that value as a positive value to determine the area between a curve and the x-axis. Sketching the graph will show what part of the function lies below the x-axis. So

$$A_{\text{curve}} = \int_a^b f(x)\,\mathrm{d}x + \left|\int_b^c f(x)\,\mathrm{d}x\right|$$

or

$$A_{\text{curve}} = \int_a^c |f(x)|\,\mathrm{d}x$$

Area between two curves



Using definite integrals you can also find the areas enclosed between curves:

$$A_{\text{between}} = \int_a^b \left(g(x) - f(x)\right)\mathrm{d}x$$

With $g(x)$ as the “top” function (furthest from the x-axis). For the area between curves, it does not matter what is above/below the x-axis.

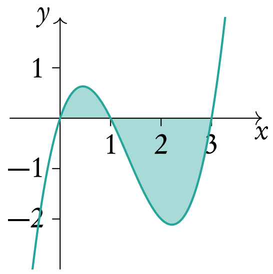
Finding areas with definite integrals.

Let $y = x^3 - 4x^2 + 3x$
Find the area from $x = 0$ to $x = 3$.

1. Find the x-intercepts: $f(x) = 0$

$x^3 - 4x^2 + 3x = 0$, using the GDC:
$x = 0$ or $x = 1$ or $x = 3$

2. If any of the x-intercepts lie within the range, sketch the function to see which parts lie above and below the x-axis.



3. Setup integrals and integrate

Left: $\int_0^1 x^3 - 4x^2 + 3x \, dx =$

$$= \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_0^1$$

$$= \left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right) - (0)$$

$$= \frac{5}{12}$$

Right: $\int_1^3 x^3 - 4x^2 + 3x \, dx =$

$$= \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_1^3$$

$$= \left(\frac{1}{4}(3)^4 - \frac{4}{3}(3)^3 + \frac{3}{2}(3)^2\right)$$

$$- \left(\frac{1}{4}(1)^4 - \frac{4}{3}(1)^3 + \frac{3}{2}(1)^2\right)$$

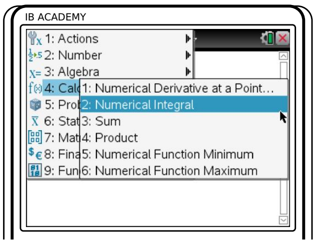
$$= -\frac{8}{3}$$

4. Add up the areas (and remember areas are never negative!)

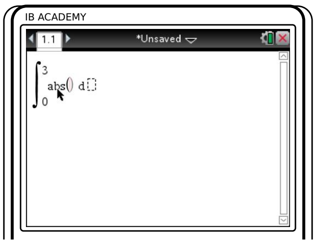
$$\frac{5}{12} + \frac{8}{3} = \frac{37}{12}$$

Alternatively, use the calculator to find areas

Calculate the area between $\int_0^3 x^3 - 4x^2 + 3x$ and the x-axis

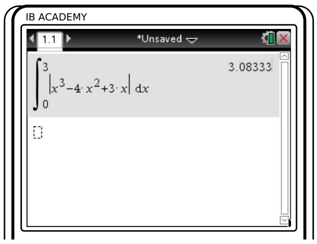


Press menu
`4: Calculus`
`2: Numerical integral`



Enter the boundaries and before putting the function.

Press ∞β°, choose
'`abs(`'



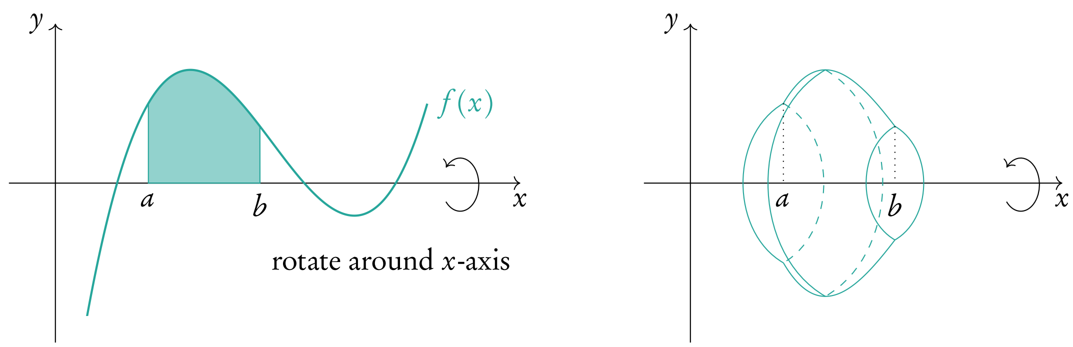
Enter the function and place the variable (usually x) after d

In this case, the area is 3.083

7.2.2 Volume of revolution

Besides finding areas under and between curves, integration can also be used to calculate the volume of the solid that a curve would make if it were rotated 360° around its axis — this is called the volume of revolution. DB 6.5

$$V = \pi \int_a^b y^2 \, dx \quad \equiv \quad V = \int_a^b \pi y^2 \, dx$$



Example.

Find the area from $x = 1$ to $x = 4$ for the function $y = \sqrt{x}$.

$$A = \int_1^4 \sqrt{x}\,\mathrm{d}x = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4 = \left[\frac{2}{3}(4)^{\frac{3}{2}}\right] - \left[\frac{2}{3}(1)^{\frac{3}{2}}\right] = \frac{14}{3}$$

This area is rotated $360°$ $(= 2\pi)$ around the x-axis. Find the volume of the solid.

$$V = \pi \int_1^4 \sqrt{x}^2\,\mathrm{d}x = \pi \int_1^4 x\,\mathrm{d}x = \pi\left[\frac{1}{2}x^2\right]_1^4 = \pi\left(\left[\frac{1}{2}(4)^2\right] - \left[\frac{1}{2}(1)^2\right]\right) = \frac{15\pi}{2}$$

PROBABILITY

8

Table of contents & cheatsheet

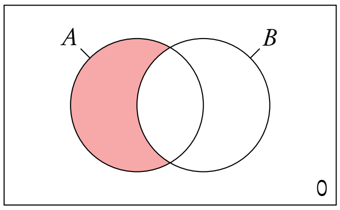
Definitions

Sample space the list of all possible outcomes.

Event the outcomes that meet the requirement.

Probability for event A, $P(A) = \dfrac{\text{Number of ways } A \text{ can happen}}{\text{all outcomes in the sample space}}$.

8.1. Single events 70



8.2. Multiple events 72

Probabilities for successive events can be expressed through tree diagrams.
In general, if you are dealing with a question that asks for the probability of:

- one event **and** another, you **multiply**
- one event **or** another, you **add**

Conditional probability used for successive events that come one after another (as in tree diagrams).

The probability of A, given that B has happened: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

8.3. Distributions 74

Expected values

$$E(X) = \sum xP(X = x)$$

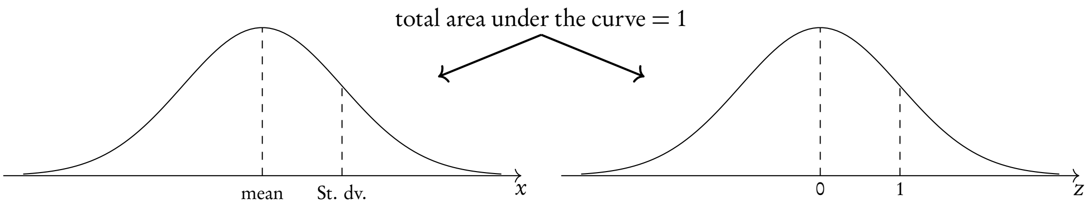
Binomial distribution with parameters n and p it should be used for situations with only 2 outcomes and lots of trials

$$P(X = x) = \binom{n}{r} p^r (1-p)^{n-r}$$

where $\binom{n}{r} = \text{nCr} = \dfrac{n!}{r!(n-r)!}$, $n =$ number of trials, $p =$ probability of success, $r =$ number of success.

On calculator:

- Binompdf(n,p,r) $P(X = r)$
- Binomcdf(n,p,r) $P(x \leq r)$
- Mean np
- Varience npq



Notation: $X \sim N(\mu, \sigma^2)$

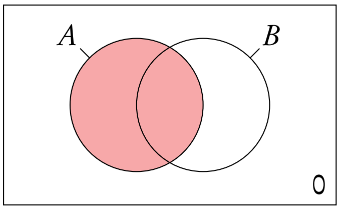
Transform to standardised normal distribution: $Z = \dfrac{x - \mu}{\sigma}$

On calculator: normal cdf (lower bound, upper bound, mean ($= \mu$), standard deviation ($= \sigma$))

ACADEMY

8.1 Single events (Venn diagrams)

Probability for single events can be expressed through venn diagrams.



Sample space the list of all possible outcomes.

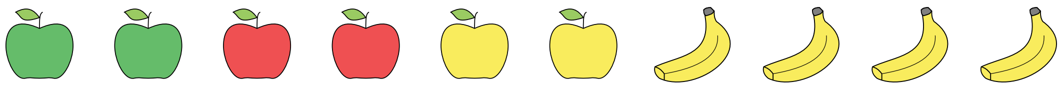
Event the outcomes that meet the requirement.

Probability for event A,

$$P(A) = \frac{\text{Number of ways } A \text{ can happen}}{\text{all outcomes in the sample space}}$$

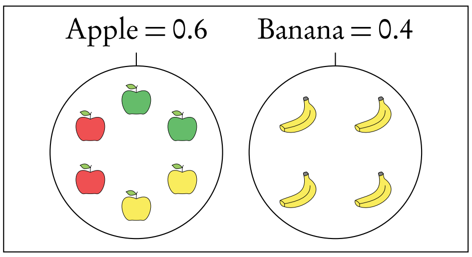
Here the shaded circle.

Imagine I have a fruit bowl containing 6 apples and 4 bananas.



Example.

I pick a piece of fruit.
What is the probability of picking each fruit?

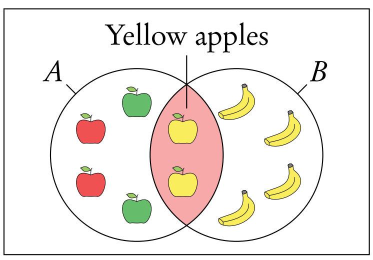


In independent events $P(A \cap B) = P(A) \times P(B)$. It will often be stated in questions if events are independent.

As apples cannot be bananas this is mutually exclusive, therefore $P(A \cup B) = P(A) + P(B)$ and $P(A \cap B) = 0$. It is also an exhaustive event as there is no other options apart from apples and bananas. If I bought some oranges the same diagram would then be not exhaustive (oranges will lie in the Sample Space).

Example.

Of the apples 2 are red, 2 are green and 2 are yellow.
What is the probability of picking a yellow apple?



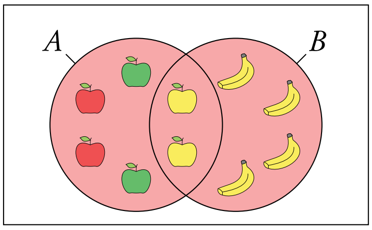
A: apples

B: yellow fruit

This is not mutually exclusive as both apples and bananas are yellow fruits. Here we are interested in the intersect $P(A \cap B)$ of apples and yellow fruit, as a yellow apple is in both sets $P(A \cap B) = P(A) + P(B) - P(A \cup B)$.

Example.

What is the probability of picking an apple or a yellow fruit?



A: apples

B: yellow fruit

This is a union of two sets: apple and yellow fruit.

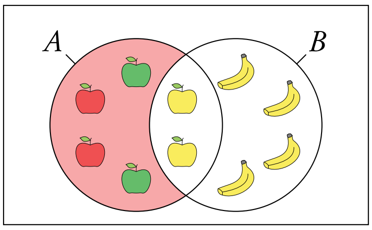
When an event is exhaustive the probability of the union is 1.

The union of events A and B is:

- when A happens;
- when B happens;
- when both A and B happen $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Example.

What is the probability of not picking a yellow fruit?



A: apples

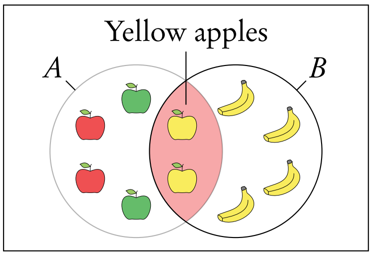
B: yellow fruit

This is known as the compliment of B or B'. $B' = 1 - B$.

Here we are interested in everything but the yellow fruit.

Example.

What is the probability of picking an apple given I pick a yellow fruit?

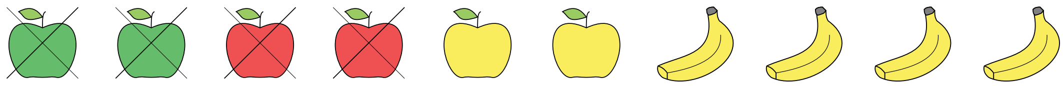


A: apples

B: yellow fruit

This is "conditional" probability in a single event. **Do not use the formula in the formula booklet.** Here we are effectively narrowing the sample space $= \dfrac{0.2}{(0.2+0.4)} = \dfrac{1}{3}$.

You can think of it like removing the non yellow apples from the fruit bowl before choosing.



Conditional probability $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

8.2 Multiple events (tree Diagrams)

Questions involving dependent events will often involve elements that are drawn "without replacement". Remember that the probabilities will be changing with each new set of branches.

Dependent events two events are dependent if the outcome of event A affects the outcome of event B so that the probability is changed.

Independent events two events are independent if the fact that A occurs does not affect the probability of B occurring.

Conditional probability used for successive events that come one after another (as in tree diagrams). The probability of A, given that B has happened: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

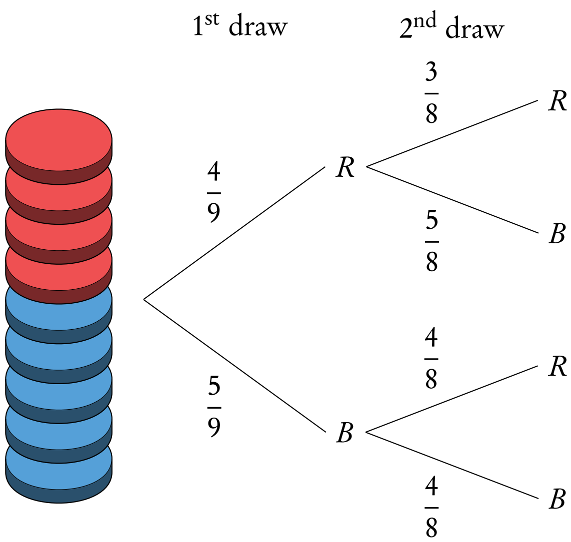
Probabilities for successive events can be expressed through tree diagrams. In general, if you are dealing with a question that asks for the probability of:

- one event **and** another, you **multiply**
- one event **or** another, you **add**

Example.

Two disks are randomly drawn without replacement from a stack of 4 red and 5 blue disks. Draw a tree diagram for all outcomes.

The probability of drawing two red disks can be found by multiplying both probabilities of getting red $\left(\frac{4}{9} \times \frac{3}{8}\right)$.



R and R: $\frac{4}{9} \times \frac{3}{8} = \frac{12}{72}$

R and B: $\frac{4}{9} \times \frac{5}{8} = \frac{20}{72}$

B and R: $\frac{5}{9} \times \frac{4}{8} = \frac{20}{72}$

B and B: $\frac{5}{9} \times \frac{4}{8} = \frac{20}{72}$

The probabilities for each event should always add up to 1. The probabilities describing all the possible outcomes should also equal 1 (that is, the probabilities that we found by multiplying along the individual branches).

What is the probability to draw one red and one blue disk?

P(one red **and** one blue)

$(P(R) \text{ and } P(B))$ **or** $(P(B) \text{ and } P(R))$

$(P(R) \times P(B))$ $\quad$ $(P(B) \times P(R))$

$$\frac{20}{72} + \frac{20}{72} = \frac{40}{72} = \frac{5}{9}$$

It is common for conditional probability questions to relate to previous answers.

What is the probability to draw at least one red disk?

P(**at least** one red)

$$P(R \text{ and } R) + P(B \text{ and } R) + P(R \text{ and } B) = 1 - P(B \text{ and } B)$$

$$\frac{12}{72} + \frac{20}{72} + \frac{20}{72} = 1 - \frac{20}{72} = \frac{52}{72} = \frac{13}{18}$$

What is the probability of picking a blue disc given that at least one red disk is picked?

$$P(\text{blue disk} \mid \text{at least one red disk}) = \frac{P(\text{a blue disk})}{P(\text{at least one red disk})} = \frac{\frac{5}{9}}{\frac{13}{18}} = \frac{10}{13}$$

Another way of dealing with multiple events is with a sample space diagram or a probability distribution.

Probability distributions.

A fair coin is tossed twice, X is the number of heads obtained.

1. Draw a sample space diagram

	H	T
H	H, H	H, T
T	T, H	T, T

2. Tabulate the probability distribution

x	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(The sum of $P(X=x)$ always equals 1)

3. Find the expected value of X: $E(X)$

$$E(X)=\sum xP(X=x)$$
$$=0\cdot\frac{1}{4}+1\cdot\frac{1}{2}+2\cdot\frac{1}{4}=1$$

So if you toss a coin twice, you expect to get heads once.

8.3 Distributions

Probability distribution a list of each possible value and their respective probabilities.

8.3.1 Distribution by function

A probability distribution can also be given by a function.

Probability distribution by function.

$P(X=x)=k\left(\frac{1}{3}\right)^{x-1}$ for $x=1,2,3$. Find constant k.

1. Use the fact that $\sum P(X=x)=1$

$$k\left(\frac{1}{3}\right)^{1-1}+k\left(\frac{1}{3}\right)^{2-1}+k\left(\frac{1}{3}\right)^{3-1}=1$$

2. Simplify and solve for k

$k+\frac{1}{3}k+\frac{1}{9}k=\frac{13}{9}k=1$. So, $k=\frac{9}{13}$.

8.3.2 Binomial distribution

Binomial distribution with parameters n and p it should be used for situations with only 2 outcomes and lots of trials

$$P(X=x)=\binom{n}{r}p^r\,(1-p)^{n-r}$$

where $\binom{n}{r} = \text{nCr} = \dfrac{n!}{r!(n-r)!}$

$n =$ number of trials

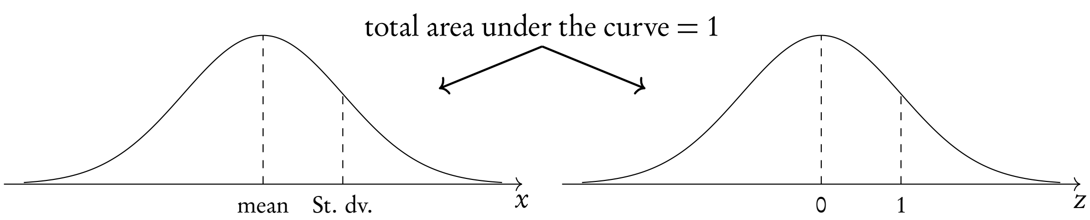
$p =$ probability of success

$r =$ number of success

8.3.3 Normal distribution

A normal distribution is one type of probability distribution which gives a bell-shape curve if all the values and their corresponding probabilities are plotted.

We can use normal distributions to find the probability of obtaining a certain value or a range of values. This can be found using the area under the curve; the area under the bell-curve between two x-values always corresponds to the probability for getting an x-value in this range. The total area under the normal distribution is always 1; this is because the total probability of getting any x-value adds up to 1 (or, in other words, you are 100% certain that your x-value will lie somewhere on the x-axis below the bell-curve).



Even though you will be using your GDC to find probabilities for normal distributions, it's always *very* useful to draw a diagram to indicate for yourself (and the examiner) what area or x-value you are looking for.

Notation: $X \sim N(\mu, \sigma^2)$

Transform to standard N: $Z = \dfrac{x-\mu}{\sigma}$

On calculator: `normal cdf` (lower bound, upper bound, mean ($=\mu$), standard deviation ($=\sigma$))

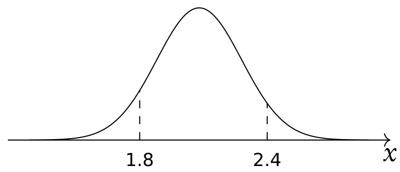
Finding mean and standard deviation of a normal distribution.

All nails longer than $2.4\,\text{cm}$ (5.5%) and shorter than $1.8\,\text{cm}$ (8%) are rejected. What is the mean and standard deviation length?

1. Write down equations

$$P(L < 1.8) = 0.08$$
$$P(L > 2.4) = 0.055$$

2. Draw a sketch!



3. Write standardized equation of the form $P(Z < \ldots)$

$$P\left(Z < \frac{1.8-\mu}{\sigma}\right) = 0.08$$
$$P\left(Z > \frac{2.4-\mu}{\sigma}\right) = 0.055$$
$$P\left(Z < \frac{2.4-\mu}{\sigma}\right) = 1 - 0.055 = 0.945$$

4. Use "`inVnorm`" on calculator

$$\texttt{inVnorm}\,(0.08, 0, 1) = -1.4051$$
$$\texttt{inVnorm}\,(0.945, 0, 1) = 1.5982$$

5. Equate and solve

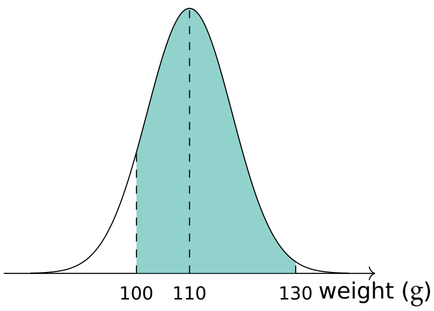
$$\begin{cases} \dfrac{1.8-\mu}{\sigma} = -1.4051 \\ \dfrac{2.4-\mu}{\sigma} = 1.5982 \end{cases}$$
$$\begin{cases} \mu = 2.08 \\ \sigma = 0.200 \end{cases}$$

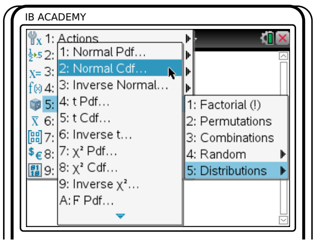
To find a probability or percentage of a whole (the area under a normal distribution curve)

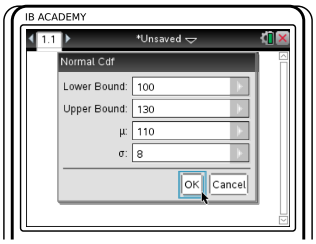
The weights of pears are normally distributed with mean $= 110\,\text{g}$ and standard deviation $= 8\,\text{g}$.
Find the percentage of pears that weigh between $100\,\text{g}$ and $130\,\text{g}$

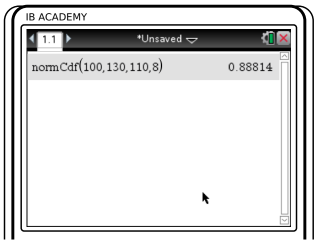
Sketch!
Indicate:

- The mean = $110\,\text{g}$
- Lower bound = $100\,\text{g}$
- Upper bound = $130\,\text{g}$
- And shade the area you are looking to find.









Press menu, choose
`5: Probability`
`5: Distributions`
`2: Normal Cdf`

Enter lower and upper boundaries, mean (μ) and standard deviation (σ).
For lower bound $= -\infty$,
  set `lower: -1E99`
For upper bound $= \infty$,
  set `upper: 1E99`

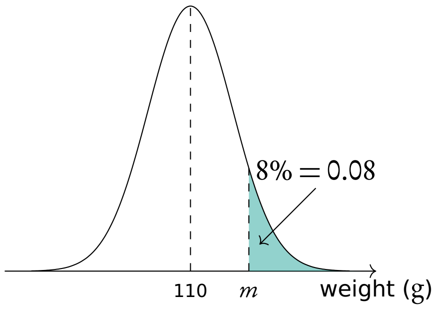
Press OK

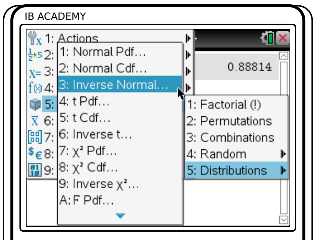
So 88.8% of the pears weigh between $100\,\text{g}$ and $130\,\text{g}$.

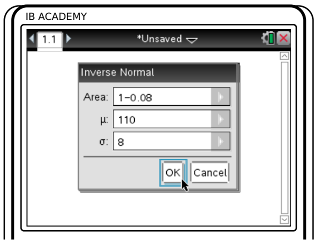
To find an x-value when the probability is given

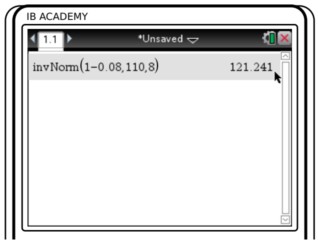
The weights of pears are normally distributed with mean $= 110$ g and standard deviation $= 8$ g. 8% of the pears weigh more than m grams. Find m.

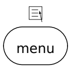
Sketch!









Press 

`5: Probability`
`5: Distributions`
`3: Inverse Normal`

Enter probability (`Area`), mean (μ) and standard deviation (σ).
The calculator assumes the area is to the left of the x-value you are looking for.
So in this case:

area $= 1 - 0.08 = 0.92$

Press 

So $m = 121$, which means that 8% of the pears weigh more than 121 g.

STATISTICS

9

Table of contents & cheatsheet

Definitions

Population the entire group from which statistical data is drawn (and which the statistics obtained represent).

Sample the observations actually selected from the population for a statistical test.

Random Sample a sample that is selected from the population with no bias or criteria; the observations are made at random.

Discrete finite or countable number of possible values (e.g. money, number of people)

Continuous infinite amount of increments (e.g. time, weight)

Note: continuous data can be presented as discrete data, e.g. if you round time to the nearest minute or weight to the nearest kilogram.

9.1. Descriptive statistics 80

Mean the average value,

$$\bar{x} = \frac{\text{the sum of the data}}{\text{no. of data points}}$$

Mode the value that occurs most often

Median when the data set is ordered low to high and the number of data points is:

- odd, then the median is the middle value;
- even, then the median is the average of the two middle values.

Range largest x-value − smallest x-value

Variance $\sigma^2 = \dfrac{\sum f(x-\bar{x}^2)}{n}$ *calculator only*

Standard deviation $\sigma = \sqrt{\text{variance}}$ *calculator only*

Grouped data: data presented as an interval.
Use the midpoint as the x-value in all calculations.

Q_1 first quartile = 25^{th} percentile.

Q_2 median = 50^{th} percentile

Q_3 third quartile = 75^{th} percentile

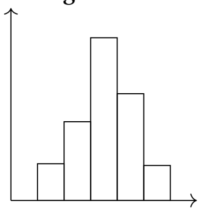
$Q_3 - Q_1$ interquartile range (IQR) = middle 50 percent

9.2. Statistical graphs 82

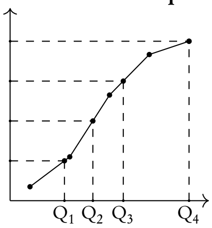
Frequency the number of times an event occurs in an experiment

Cumulative frequency the sum of the frequency for a particular class and the frequencies for all the classes below it

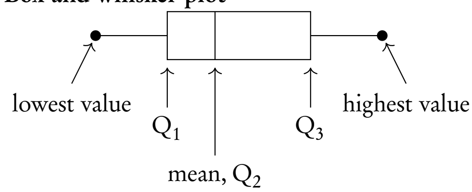
Histogram



Cumulative frequency



Box and whisker plot



9.3. Bi-variate analysis 86

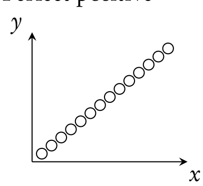
Interpretation of r-values

r-value	correlation
$0.00 \le \lvert r \rvert \le 0.25$	very weak
$0.25 \le \lvert r \rvert \le 0.50$	weak
$0.50 \le \lvert r \rvert \le 0.75$	moderate
$0.75 \le \lvert r \rvert \le 1.00$	strong

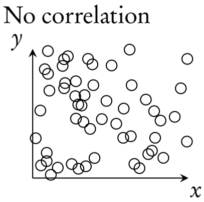
Correlation does not mean causation.

Scatter diagrams

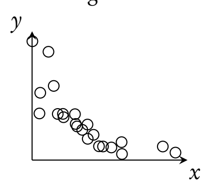
Perfect positive



No correlation



Weak negative



9.1 Descriptive statistics

The mean, mode and median, are all ways of measuring "averages". Depending on the distribution of the data, the values for the mean, mode and median can differ slightly or a lot. Therefore, the mean, mode and median are all useful for understanding your data set.

Example data set: 6, 3, 6, 13, 7, 7 in a table:

x	3	6	7	13
frequency	1	2	2	1

Mean the average value, $\bar{x} = \dfrac{\text{the sum of the data}}{\text{no. of data points}} = \dfrac{\sum x}{n} = \dfrac{\sum f x}{\sum f}$

e.g. $\bar{x} = \dfrac{3+6+6+7+7+13}{6} = \dfrac{1\cdot 3+2\cdot 6+2\cdot 7+1\cdot 13}{1+2+2+1} = 7$

Mode the value that occurs most often (highest frequency)

e.g. The example data set has 2 modes: 6 and 7

Median the middle value when the data set is ordered low to high. Even number of values: the median is the average of the two middle values. Find for larger values as $n + \dfrac{1}{2}$.

e.g. data set from low to high: 3, 6, 6, 7, 7, 13

median $= \dfrac{6+7}{2} = 6.5$

Range largest x-value $-$ smallest x-value

e.g. range $= 13 - 3 = 10$

Variance $\sigma^2 = \dfrac{\sum f(x - \bar{x}^2)}{n}$ *calculator only*

Standard deviation $\sigma = \sqrt{\text{variance}}$ *calculator only*

Note on grouped data: data presented as an interval; e.g. 10–20 cm.

- Use the midpoint as the x-value in all calculations. So for 10–20 cm use 15 cm.
- For 10–20 cm, 10 is the lower boundary, 20 is the upper boundary and the width is $20 - 10 = 10$.

Adding a constant to all the values in a data set or multiplying the entire data set by a constant influences the mean and standard deviation values in the following way:

Table 9.1: Adding or multiplying by a constant

	adding constant k	**multiplying by k**
mean	$\bar{x} + k$	$k \times \bar{x}$
standard deviation	σ	$k \times \sigma$

Q_1	first quartile	$= 25^{th}$ percentile.
	The value for x so that 25% of all the data values are $\leq$ to it.	
Q_2	median	$= 50^{th}$ percentile
Q_3	third quartile	$= 75^{th}$ percentile
$Q_3 - Q_1$	interquartile range (IQR)	= middle 50 percent

Example.

Snow depth is measured in centimeters:
30, 75, 125, 55, 60, 75, 65, 65, 45, 120, 70, 110.
Find the range, the median, the lower quartile, the upper quartile and the interquartile range.

First always rearrange data into ascending order: 30, 45, 55, 60, 65, 65, 70, 75, 75, 110, 120, 125

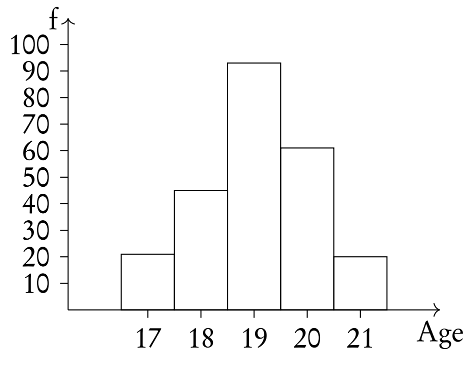
1. The range:
$$125 - 30 = 95\,\text{cm}$$
2. The median: there are 12 values so the median is between the 6^{th} and 7^{th} value.
$$\frac{65 + 70}{2} = 67.5\,\text{cm}$$
3. The lower quartile: there are 12 values so the lower quartile is between the 3^{rd} and 4^{th} value.
$$\frac{55 + 60}{2} = 57.5\,\text{cm}$$
4. The upper quartile: there are 12 values so the lower quartile is between the 9^{th} and 10^{th} value.
$$\frac{75 + 110}{2} = 92.5\,\text{cm}$$
5. The IQR
$$92.5 - 57.5 = 35\,\text{cm}$$

9.2 Statistical graphs

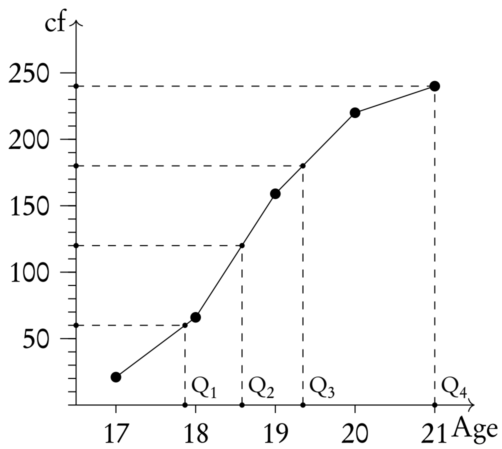
Frequency the number of times an event occurs in an experiment

Cumulative frequency the sum of the frequency for a particular class and the frequencies for all the classes below it

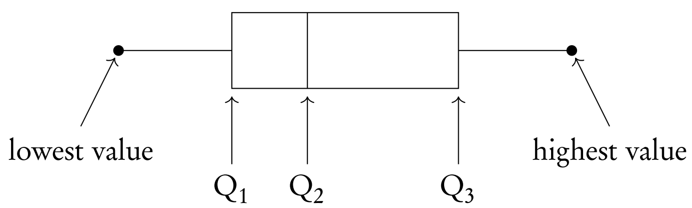
Age	17	18	19	20	21
No. of students	21	45	93	61	20
Cumulative freq.	21	66	159	220	240



A histogram is used to display the frequency for a specific condition. The frequencies (here: # of students) are displayed on the y-axis, and the different classes of the sample (here: age) are displayed on the x-axis. As such, the differences in frequency between the different classes assumed in the sample can easily be compared.



The cumulative frequency graph is used to display the development of the frequencies as the classes of the event increase. The graph is plotted by using the sum of all frequencies for a particular class, added to the frequencies for all the classes below it. The classes of the event (age) are displayed on the x-axis, and the frequency is displayed on the y-axis. The cumulative frequency graph always goes upwards, because the cumulative frequency increases as you include more classes.



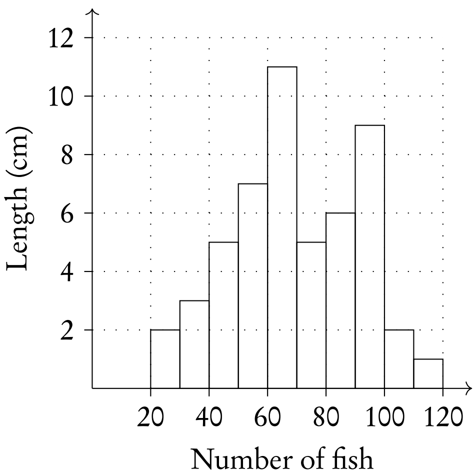
Box and whisker plots neatly summarize the distribution of the data. It gives information about the range, the median and the quartiles of the data. The first and third quartiles are at the ends of the box, the median is indicated with a vertical line in the interior of the box, and the maximum and minimum points are at the ends of the whiskers.

Outliers will be any points lower than $Q_1 - 1.5 \times IQR$ and larger than $Q_3 + 1.5 \times IQR$ (IQR = interquartile range)

To identify the value of Q_1, Q_2 and Q_3, it is easiest to use the cumulative frequency graph. First, determine the percentage of the quartile in question. Second, divide the total cumulative frequency of the graph (i.e. the total sample size) by 100 and multiply by the corresponding percentage. Then, you will have found the frequency (y-value) at which 25% for Q_1 / 50% for Q_2 / 75% for Q_3 of the sample is represented. To find the x-value, find the corresponding x-value for the previously identified y-value.

Example.

Using the histogram, create a cumulative frequency graph and use it to construct a box and whisker diagram.

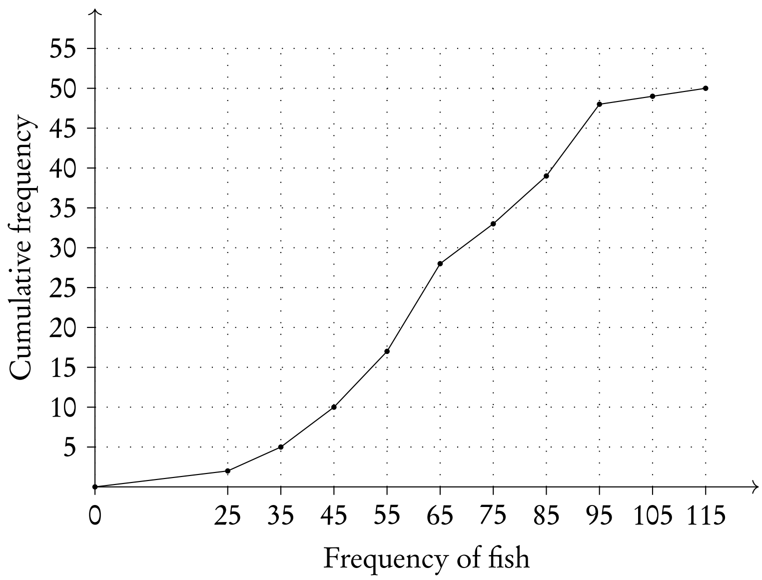


Write out the table for frequency and cumulative frequency.

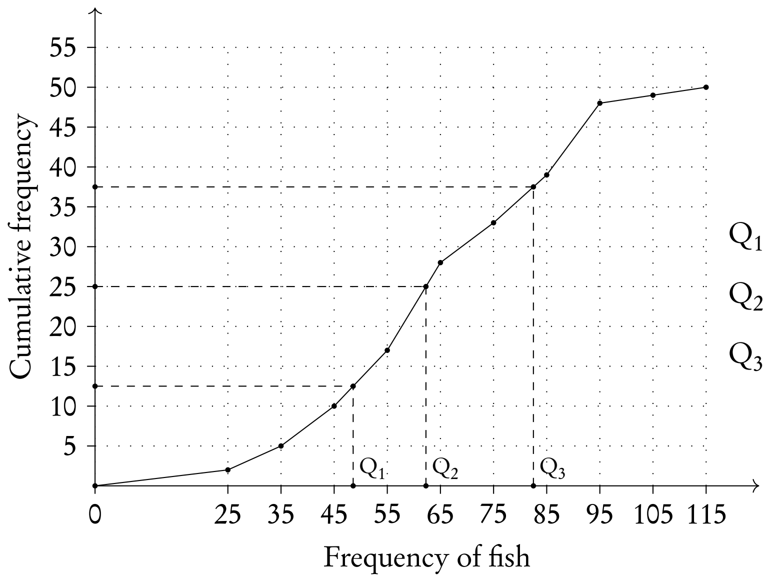
Frequency of fish	20–30	30–40	40–50	50–60	60–70	70–80	80–90	90–100	100–110	110–120
Length of fish	2	3	5	7	11	5	6	9	1	1
Cumulative f.	2	5	10	17	28	33	39	48	49	50

Example.

Plot on cumulative frequency chart. Remember to use the midpoint of the date, e.g., 25 for 20–30.



Use graph to find Q_1, Q_2 and Q_3.

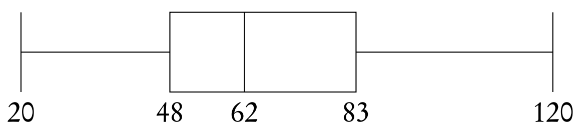


Q_1 25% of $50 = 12.5 \rightarrow 48$

Q_2 50% of $50 = 25 \rightarrow 62$

Q_3 75% of $50 = 37.5 \rightarrow 83$

Plot box and whiskers.

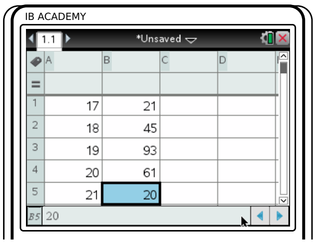


ACADEMY

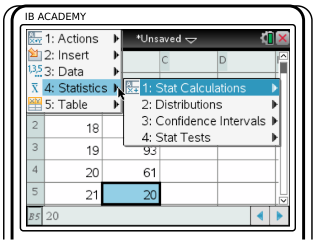
GDC

To find mean, standard deviation and quartiles etc.

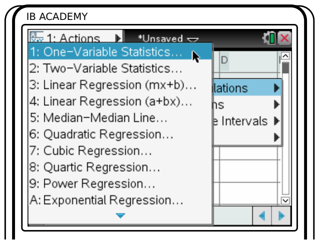
For the data used in the previous example showing the ages of students



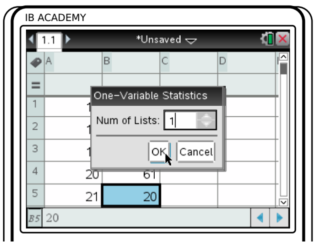
Press on, go to `Lists and Spreadsheets`. Enter x-values in `L1` and, if applicable, frequencies in `L2`



Press menu, choose
`4: Statistics`
`1: Stat Calculations`

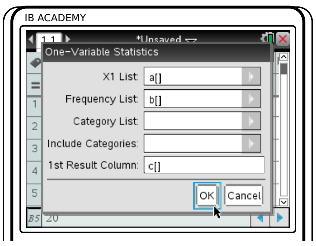


`1: One-Variable Statistics`



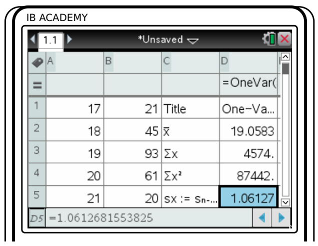
Enter `Num of lists: 1`.

Press OK



Enter names of columns you used to enter your x-list and frequency list and column where you would like the solutions to appear: `a[]`, `b[]` and `c[]`.

Press OK



mean = 19.06;
standard deviation = 1.06
etc.

9.3 Bi-variate analysis

Bi-variate analysis is a method of assessing how two (bi) sets of data (variables) correlate to one another. We use Pearson's correlation to put a number to this relationship

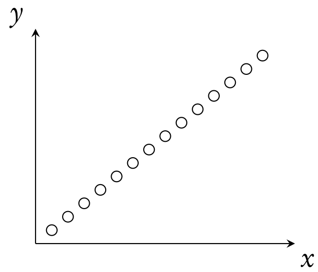
Pearson's correlation r is a measure to assess the linear correlation between two variables, where 1 is total positive correlation, 0 is no correlation, and −1 is total negative correlation.

Interpretation of r-values:

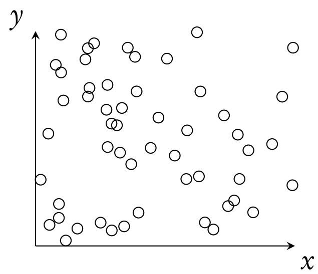
r-value	$0 \leq \lvert r \rvert \leq 0.25$	$0.25 \leq \lvert r \rvert \leq 0.50$	$0.50 \leq \lvert r \rvert \leq 0.75$	$0.75 \leq \lvert r \rvert \leq 1$
correlation	very weak	weak	moderate	strong

Scatter diagrams

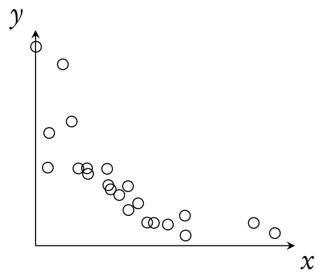
Perfect positive correlation $r = 1$



No correlation $r = 0$



Weak negative correlation $-0.5 < r < -0.25$



However it is important to remember this maxim: *Correlation does not mean causation*.

Just because two variables have a relationship it does not mean they cause one another. For example Ice cream sales show a strong correlation to the number deaths by drowning. Therefore we might falsely state ice cream consumption causes drowning. But it is more plausible that both are caused by warm weather leading to more desire for ice cream and swimming and are just correlated.

Using GDC

Calculate by finding the regression equation on your GDC: make sure `STAT DIAGNOSTICS` is turned `ON` (can be found when pressing `MODE`).

Bivariate statistics can also be used to predict a mathematical model that would best describe the relationship between the two measured variables; this is called regression. Here you will only have to focus on linear relationships, so only straight line graphs and equations.

Your 'comment' on Pearson's correlation always has to include two things:

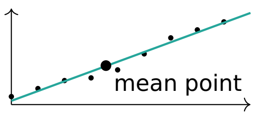
1. Positive / negative *and*
2. Stong / moderate / weak / very weak

Find Pearson's correlation r and comment on it

The height of a plant was measured the first 8 weeks

Week x	0	1	2	3	4	5	6	7	8
Height (cm) y	23.5	25	26.5	27	28.5	31.5	34.5	36	37.5

1. Plot a scatter diagram



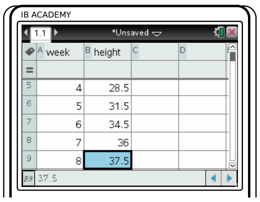
2. Use the mean point to draw a best fit line

$$\bar{x} = \frac{0+1+2+\ldots+8}{9} = 3.56$$

$$\bar{y} = \frac{23.5+25+\ldots+37.5}{9} = 30$$

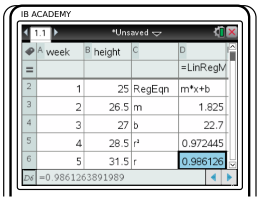
3. Find the equation of the regression line Using GDC

$y = 1.83x + 22.7$

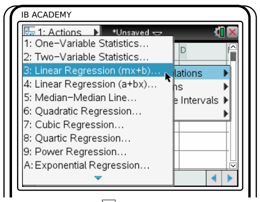


Press on (off), got to "`Lists and Spreadsheets`"

Enter x-values in one column (e.g A) and y-values in another column (e.g. B)



So, equation of regression line is $y = 1.83x + 22.7$ and Pearson's correlation (r-value) $= 0.986$

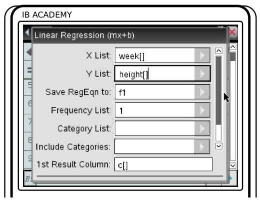


Press menu

`4: Statistics`

`1: Stat Calculations`

`3: Linear Regression (mx+b)`



Enter

`X list: A [];`

`Y list: B[];`

`1st Result Column: C[]`

Press OK

4. Comment on the result.

Pearson's correlation is $r = 0.986$, which is a strong positive correlation.